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Fuzzy boolean programming problems with fuzzy costs: A general study

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Abstract

This paper deals with boolean linear programming problems involving coefficients in the objective function as fuzzy numbers. In the study of these problems different approaches can be proposed to use ranking fuzzy numbers methods and fuzzy preference relations obtaining auxiliary classical boolean programming problems, and to use the representation theorem obtaining a convex set with extreme points defined by the lower or upper bound of the α-level sets of the fuzzy coefficients, and consequently an auxiliary interval boolean programming problem. In this paper we develop and link the different approaches.

Keywords: Mathematical programming; Fuzzy numbers; Fuzzy boolean programming; Fuzzy objectives

1. Introduction

Boolean linear programming problems deal with problems of maximizing or minimizing a function of many variables subject to inequality and equality constraints and integrality restrictions on some or all of the variables (boolean variables). Due to the robustness of the general model, a wide variety of problems can be represented by this model. Applications in many fields, such as those related to operations research and management (knapsack, assignment, matching, covering, facility location, network flow, etc.) [36, 38] artificial intelligence (modeling propositional logic, reasoning, etc.) [5, 11, 20, 25, 43, 44, 47], etc.

To have some difficulties when modeling a real world problem by means of a boolean programming problem is normal. One of such difficulties is either in the fact that goals and constraints are often represented by the vague linguistic form or in the fact that the parameters are not known exactly.

Often in a real world problem a decision-maker or an expert gives approximate estimates about the true values of the objective coefficients rather than the exact values of these, moreover those estimates can be given with some vagueness. As was observed in [29]:

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"In practice, the unit costs/profits of new products or new projects, lending and borrowing interest rates, and cash flows are always imprecise".

To consider the costs as fuzzy numbers is the most reasonable way to model the former discrete optimization problem. Then the problem may be formulated as follows

$$\begin{aligned} \max \quad & z = \sum_{j \in N} \underset{\sim}{c}_j x_j \\ \text{s.t.} \quad & \sum_{j \in N} a_{ij} x_j \leqslant b_i, \quad i \in M, \\ & x_j \in \{0, 1\}, \quad j \in N, \end{aligned} \tag{1}$$

a_{ij}, $b_i \in \mathbb{R}$, $i \in M$, $j \in N$ and $\underset{\sim}{c}_j \in F(\mathbb{R})$, where $F(\mathbb{R})$ is the set of fuzzy numbers. Thus one has the membership functions

$$\mu_j : \mathbb{R} \to (0, 1], \quad j \in N$$

expressing the lack of precision on the values of the coefficients that the decision-maker has.

Fuzzy discrete programming models have been studied in various publications. A classification of them was shown in [13], different models, methods and applications have been presented [1, 10, 14, 18, 22, 27, 35, 39, 48, 49], also some approaches to fuzzy boolean linear programming (FBLP) problems with fuzzy costs have been described [8, 9].

This paper is devoted to look further into the different approaches to FBLP problems with fuzzy costs. The first one consists of the use of several well known ranking fuzzy numbers methods, each of them provides a different auxiliary conventional optimization model solving the former problem. The second one explores the behavior of the representation theorem for fuzzy sets when used as tool to solve the proposed problem. The link between the auxiliary models will be finally studied.

In order to do this the paper is set out as follows. In Section 2 the formulation of the problem is presented. Section 3 is addressed to the use of ranking fuzzy numbers methods, methods of optimal alternatives, ranking functions and fuzzy relations, along with a description of the linkage in the use of ranking functions and fuzzy relations. Section 4 is devoted to relating the approach using the representation theorem, solutions methods (with efficient points and weight vector, and with the interval arithmetic) and the link between them is shown. In Section 5 a numerical example is analyzed, and some interesting conclusions are finally pointed out.

2. Formulation of the problem

The overall situation corresponding to a FBLP problem with fuzzy costs can be described by the following example.

Illustrative example. In a Faculty of Computer Science, one wishes to buy the equipment for some computer rooms. Each room will have different equipment in order to include a great variety of work-stations. Six different types of proposals are received and a study is carried out based on the number of students that will use the equipment. As seen in the following table each number of students is given as a percentage of total students and the cost of each class-room is given in millions of pesetas.

Type of class	Cost	Percentage of students
A	14	55
B	11	40
C	17	50
D	7	28
E	13	35
F	10	43

the percentages of use can vary up to 8%, 5%, 12%, 10%, 7% and 12%, respectively.

The goal is to purchase equipment for the classrooms so that they will be used by the maximum possible number of students. Thirty-two million pesetas are available in order to buy the equipment, it is only possible to buy equipment for three classrooms and it is necessary to have at least one of type A, B or C, and another of type C, E or F.

In this problem the constraints are given in a precise form and the objective is imprecise. The percentage of use of every classroom can be represented by triangular membership functions.

The formulation of this problem can be written as

$$\begin{aligned} \max \quad & \underset{\sim}{55}x_1 + \underset{\sim}{40}x_2 + \underset{\sim}{50}x_3 + \underset{\sim}{28}x_4 + \underset{\sim}{35}x_5 + \underset{\sim}{43}x_6 \\ \text{s.t.} \quad & 14x_1 + 11x_2 + 17x_3 + 7x_4 + 13x_5 + 10x_6 \leqslant 32, \\ & x_1 + x_2 + x_3 \geqslant 1, \\ & x_3 + x_5 + x_6 \geqslant 1, \\ & x_j \in \{0,1\}, \quad 1 \leqslant j \leqslant 6, \end{aligned}$$

where the fuzzy costs are the following triangular fuzzy numbers

$$\underset{\sim}{c}_1 = (47,55,63), \qquad \underset{\sim}{c}_2 = (35,40,45), \qquad \underset{\sim}{c}_3 = (38,50,62),$$

$$\underset{\sim}{c}_4 = (18,28,38), \qquad \underset{\sim}{c}_5 = (28,35,42), \qquad \underset{\sim}{c}_6 = (31,43,55).$$

Next, the different approaches are studied. The above problem will be solved in Section 5.

3. Approach using methods of ranking fuzzy numbers

Let X be the finite set of feasible solutions to (1),

$$X = \{x/Ax \leqslant b,\ x \in \{0,1\}^n\} \tag{2}$$

and g be the function mapping the set of alternatives into the set of fuzzy numbers,

$$g : X \to F(\mathbb{R})$$

defined by

$$g(x) = \underset{\sim}{c}x = \sum_{i \in S} \underset{\sim}{c}_i, \quad S = \{i \in N/x_i = 1\}, \qquad \underset{\sim}{c}_i \in F(\mathbb{R}) \ \forall i \in S. \tag{3}$$

An alternative $x' \in X$ is called optimal if the fuzzy number $g(x')$ is the greatest in the set of fuzzy numbers

$$A = \{g(x)/x \in X\},$$

l being the number of elements of A, and L the indexset. In order to determine the greatest fuzzy number of the finite set $A = \{A^1, \ldots, A^l\} \subset F(\mathbb{R})$ we have three possible methods.

1. The first is the method of optimal alternatives, in the following form:

Given $A = \{A^i/i \in L\}$, the set of fuzzy numbers, where S_i is the support of the fuzzy number A^i,

$$S_i = \operatorname{Supp} \underset{\sim}{A}^i = \{x \in \mathbb{R}/\mu_{A^i}(x) > 0\} \subset I$$

in which, without loss of generality, one supposes that the domain of definition is in the interval I.

The aim is to obtain a fuzzy set of optimal alternatives

$$\tilde{O} = \{i, \mu_{\tilde{O}}(i)\}, \quad i \in T,$$

where $\mu_{\tilde{O}}(i)$ is the degree to which the ith alternative may be considered the best alternative. Some methods of optimal alternatives are [4, 50]: method of Jain, method of Chen, method of Baas and Kwakernaak, method of Baldwin and Guild, etc.

2. The second method consists of the definition of a ranking function, f, mapping each fuzzy set into the real line, $\mathbb{R}$, where a natural order exists.

So called $f: A \to \mathbb{R}$, is such that

$$f(A^i) < f(A^j) \text{ implies } A^i \text{ is smaller than } A^j,$$

$$f(A^i) > f(A^j) \text{ implies } A^i \text{ is greater than } A^j,$$

$$f(A^i) = f(A^j) \text{ implies } A^i \text{ is equal to } A^j.$$

Hence, the greatest fuzzy number, A^k, is obtained for the maximum value of the set $\{f(A^i), A^i \in A\}$, $f(A^k)$, which is associated to A^k. Some ranking functions are [4, 50]: functions of Yager, method of Chang, Adamo's index, average index, etc.

3. The third method consists of the use of the concept of decision making with a fuzzy strict preference relation and nondominated alternatives [30]. In order to compare the elements in A, the fuzzy relation R is defined on the Cartesian product $F(\mathbb{R}) \times F(\mathbb{R})$ into the interval [0, 1].

$$\mu_R: F(\mathbb{R}) \times F(\mathbb{R}) \to [0, 1],$$

$\mu_R(A^i, A^j)$, $A^i, A^j \in A$, interpreted as a truth value of the expression "A_i is greater than or equal to A_j". This relation serves to build the fuzzy strict preference relation as follows

$$\mu_S(A^i, A^j) = \max\{0, \mu_R(A^i, A^j) - \mu_R(A^j, A^i)\},$$

we can say that A^j is strictly dominated by A^i to the degree $\mu_S(A^i, A^j)$. And, we can define the fuzzy set μ_{ND} of nondominated elements as

$$\mu_{\mathrm{ND}}(A^i) = 1 - \max_{A^j \in A} \mu_S(A^j, A^i),$$

which introduces the subset of nondominated alternatives, having the previous membership function. The value $\mu_{\mathrm{ND}}(A^i)$ is understood as a degree to which the alternative A^i is dominated by none of the elements of set A. The greatest fuzzy number belongs to the following set

$$\left\{A^i \in A/\mu_{\mathrm{ND}}(A^i) = \max_{A^j \in A} \mu_{\mathrm{ND}}(A^j)\right\}.$$

Some fuzzy preference relations are [17]: Degree of Possibility, Degree of Necessity, etc.

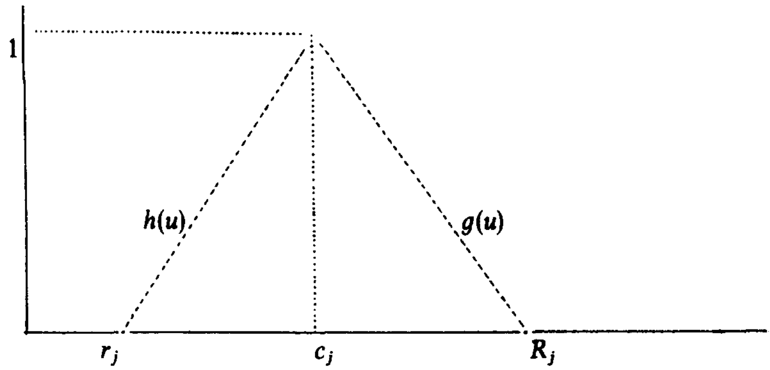
In order to obtain the solution to problem (1), we give the following definition, which depends on the method used.

Definition 1. $x^* \in X$ is an optimal solution for (1) if
(i) given a method of optimal alternatives, $\tilde{O} = \{i, \mu_{\tilde{O}}(i)\}$, then $\mu_{\tilde{O}}(g(x^*)) \geqslant \mu_{\tilde{O}}(g(x)) \forall x \in X$;
(ii) given a ranking function f then $f(g(x^*)) \geqslant f(g(x)) \forall x \in X$;
(iii) given a fuzzy relation R, and having obtained the nondominated degree $\mu_{\text{ND}}(\cdot)$ then $\mu_{\text{ND}}(g(x^*)) = \max_{x \in X} \mu_{\text{ND}}(g(x))$.

In this paper, linear triangular fuzzy numbers, $\underset{\sim}{c}_j = (r_j, c_j, R_j)$, are considered in order to simplify the operations. Its membership functions are given in the following way.

$$\forall u \in \mathbb{R},\ j \in N,\quad \mu_{c_j}(u) = \begin{cases} h(u) = (u - r_j)/(c_j - r_j), & r_j \leqslant u \leqslant c_j, \\ g(u) = (R_j - u)/(r_j - c_j), & c_j \leqslant u \leqslant R_j, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

graphically



Then the following result holds.

Proposition 2. *Let us suppose that we have linear expression $\underset{\sim}{y} = \sum_j \underset{\sim}{c}_j x_j = \underset{\sim}{c}x$ where $\underset{\sim}{c}_j$, $j = 1, \ldots, n$ are fuzzy numbers the membership function of which is similar to* (4) *and $x_j \geqslant 0$, $j \in N$. Then the membership function of the fuzzy number $\underset{\sim}{y}$ is*

$$\mu(z) = \begin{cases} (z - rx)/(cx - rx) & \text{if } rx \leqslant z \leqslant cx, \\ (Rx - z)/(Rx - cx) & \text{if } cx \leqslant z \leqslant Rx, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

where $r = (r_1, \ldots, r_n)$, $c = (c_1, \ldots, c_n)$ and $R = (R_1, \ldots, R_n)$.

The proof is obvious.

In the following we define the vectors d and d' such that $d = R - c$ and $d' = c - r$. Hence dx and $d'x$ are the lateral margins (right and left, respectively) of the fuzzy number $\underset{\sim}{c}x$.

Next, we shall apply different methods of ranking fuzzy numbers to obtain the optimal value of A. It is interesting to observe how the optimal solution obtained when a FBLP problem is solved by using a ranking method is the same as the optimal one obtained by means of a boolean programming problem with a similar constraints set and a nonfuzzy objective function.

3.1. Method of optimal alternative of Jain [24]

In the method suggested by Jain, firstly the maximum value of the support S must be determined, $Z_{\max}$, where

$$S = \bigcup_{A^i \in A} \operatorname{Supp} A^i.$$

Then the maximizing set of S is evaluated,

$$\underset{\sim}{u}_{\max} = \{z, \mu_{\max}(z)\}, \quad z \in S,$$

where

$$\mu_{\max} = \left[\frac{z}{z_{\max}}\right]^k, \quad k > 0,$$

and finally, the selected alternative verifies,

$$\mu_{\tilde{O}}(i) = \operatorname*{Sup}_z \{\mu_{A^i}(z) \wedge \mu_{\max}(z)\}.$$

In our study we consider $k = 1$. The maximum value of the support S is determined by solving the following problem:

$$\begin{aligned} \max \quad & z = Rx \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N, \end{aligned} \tag{6}$$

where $z_{\max} = Rx'$, with x' the optimal solution.

Proposition 3. *Given problem* (1), $x^* \in X$ *is the optimal solution for* (1) *using the method of Jain if* x^* *is the optimal solution for the following boolean programming problem*

$$\begin{aligned} \max \quad & T(x) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N, \end{aligned} \tag{7}$$

where

$$T(x) = \frac{Rx \cdot z_{\max} - cx(Rx - cx)}{z_{\max} \cdot z_{\max}}$$

and $z_{\max}$ *is obtained in* (6).

Proof. x^* is the solution for (1) if

$$g(x^*) = A^k \quad \text{and} \quad \mu_{\tilde{O}}(k) = \mu_{\tilde{O}}(g(x^*)) = \operatorname*{Sup}_z \{\mu_{\max}(z) \wedge \mu_{A^k}(z)\},$$

where $\mu_{\max}(z) = (z/z_{\max})$. This expression is developed in the following form. Let $g(x) = \underset{\sim}{c}x = A^k = (rx^k, cx^k, Rx^k)$, $\forall x \in X$.

If $cx^k = Rx^k$ then $\mu_{\tilde{G}}(k) = cx^k/z_{\max}$ else $\mu_{\tilde{G}}(k)$ is the value of the function $\mu_{\max}(\cdot)$, in the point obtained in the intersection of this function, $\mu_{\max}(\cdot)$, and the right side of the fuzzy number, that is, $z_0 \in \mathbb{R}$.

$$(Rx^k - z_0)/(Rx^k - cx^k) = cx^k/z_{\max},$$

$$z_0 = Rx^k - cx^k(Rx^k - cx^k)/z_{\max},$$

$$\mu_{\tilde{G}}(k) = z_0/z_{\max} = \frac{Rx^k \cdot z_{\max} - cx^k(Rx^k - cx^k)}{z_{\max} \cdot z_{\max}}.$$

Thus, we define the function

$$T(x) = \frac{Rx \cdot z_{\max} - cx(Rx - cx)}{z_{\max} \cdot z_{\max}},$$

where if $cx' = Rx'$ then $T(x') = Rx'/z_{\max}$, a similar expression obtained for $\mu_{\tilde{G}}(k)$ above.

Therefore, $\mu_{\tilde{G}}(g(x^*)) = \mathrm{Sup}_z\{\mu_{\max}(z) \wedge \mu_{A^k}(z)\}$ if $T(x^*) \geqslant T(x)\ \forall x \in X$, whereupon x^* is optimal for (1) if it is the optimal solution for (7). □

3.2. *The use of ranking functions*

Consider a ranking function f mapping each fuzzy set into the real line, $f: A \to \mathbb{R}$ we have the following result.

Proposition 4. *Given the problem* (1), $x^* \in X$ *is the optimal solution for* (1) *using a ranking function* f *if* x^* *is the solution for the following boolean programming problem*

$$\begin{aligned} \max \quad & f(\underset{\sim}{c}x) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{8}$$

Proof. As $x^* \in X$ is an optimal solution for (1) if given a ranking function f, $f(g(x^*)) \geqslant f(g(x))\ \forall x \in X$, and $g(x) = \underset{\sim}{c}x$, this is similar to solve the following problem:

$$\begin{aligned} \max \quad & f(\underset{\sim}{c}x) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \quad \square \end{aligned}$$

3.2.1. *Index of Chang* [13]

Chang defines the index function

$$f(\underset{\sim}{u}_j) = \int_{S_j} z \cdot \mu_{\underset{\sim}{u}_j}(z)\, \mathrm{d}z.$$

For triangular fuzzy numbers, it is reduced to

$$f(\underset{\sim}{c}x) = (dx + d'x)(3cx + dx - d'x)/6.$$

So that this allows us to obtain the optimal solution for (1), applying Chang's index if we solve the following boolean programming problem:

$$\begin{aligned} \max \quad & \frac{dx + d'x}{6}(3cx + dx - d'x) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{9}$$

3.2.2. Index of Yager [45, 46]

Yager has proposed several ranking functions, which we shall study next.

3.2.2.1. First index of Yager. The first function is

$$f_1(\underset{\sim}{u}_j) = \frac{\int_{S_j} g(z)\mu_{\underset{\sim}{u}_j}(z)\,\mathrm{d}z}{\int_{S_j} \underset{\sim}{\mu}_j(z)\,\mathrm{d}z},$$

where the weight $g(z)$ is a measure of the importance of the value z. If we assume linear weights, that is, $g(z) = z$, then $f_1(\underset{\sim}{u}_j)$ represents the center of gravity of the fuzzy set $\underset{\sim}{u}_j$. For triangular fuzzy numbers

$$f_1(\underset{\sim}{c}x) = cx + 1/3(dx - d'x) = (c + 1/3(d - d'))x.$$

So that this allows us to obtain the optimal solution for (1), applying the first Yager's index if we solve the following boolean programming problem

$$\begin{aligned} \max \quad & (c + 1/3(d - d'))x \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{10}$$

3.2.2.2. Second index of Yager. The second index is suggested by the possibility theory:

$$f_2(\underset{\sim}{u}_j) = \max_{z \in S_j} \min(z, \mu_{u_j}(z)).$$

In this case $f_2(\underset{\sim}{u}_j)$ measures the consistency of $\underset{\sim}{u}_j$ with the linear fuzzy set $\underset{\sim}{z}$ defined by $\mu_{\underset{\sim}{z}}(z) = z$. For triangular fuzzy numbers, it is reduced to

$$f_2(\underset{\sim}{c}x) = (cx + dx)/(dx + 1).$$

Applying the second Yager's index we obtain the optimal solution for (1), solving the following boolean programming problem

$$\begin{aligned} \max \quad & (cx + dx)/(dx + 1) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{11}$$

3.2.2.3. Third index of Yager. The third ranking function proposed by Yager is the following:

$$f_3(\underset{\sim}{u}_j) = \int_0^1 M(U_j^\alpha)\,\mathrm{d}\alpha,$$

where U_j^α is the α-level set of $\underset{\sim}{u}_j$ and $M(U_j^\alpha)$ is the mean value of the elements of U_j^α.

For triangular fuzzy numbers, it is reduced to

$$f_3(\underset{\sim}{c}x) = cx + 1/4(dx - d'x).$$

The optimal solution for (1), applying the third Yager's index is obtained solving the following boolean linear programming problem:

$$\begin{aligned} \max \quad & cx + 1/4(dx - d'x) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{12}$$

3.2.3. Relation of Adamo

Adamo uses the concept of α-level set to obtain an α-preference index which is given by

$$F_\alpha(\underset{\sim}{u}_j) = \max\{z/\mu_{u_j}(z) \geqslant \alpha\}$$

for a given threshold $\alpha \in [0, 1]$.

For triangular fuzzy numbers, it is reduced to

$$F_\alpha(\underset{\sim}{c}x) = cx + dx(1 - \alpha) \quad \forall \alpha \in [0, 1].$$

Given the problem (1), if we apply this method to solve it, we obtain the optimal solution solving the following parametric linear boolean programming problem

$$\begin{aligned} \max \quad & z(\alpha) = cx + dx(1 - \alpha) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N, \end{aligned} \tag{13}$$

where we denote this problem as $P(\alpha)$ and $x^*(\alpha)$ its optimal solution, which depends on α, the level of preference of the solution $x^*(\alpha)$ is represented by α. A method to solve (13) can be seen in [14].

3.2.4. Average index

The average value was introduced to help in the ordering of fuzzy numbers and defined by means of an integrating process of a parametric function representing the position of every α-cut in the real line.

For triangular fuzzy numbers, it is reduced to

$$V_t^\lambda(\underset{\sim}{c}x) = cx + (cx - rx)/(t + 1) + \lambda(Rx - rx)/(t + 1),$$

where the parameter λ is an optimism–pessimism degree, which must be selected by the decision-maker. When the most advantageous decision is to choose the lowest quantity ($\lambda = 0$) and an optimistic person would think of the upper quantity ($\lambda = 1$). We consider $\lambda = 1, 0.5, 0$. The parameter t is used in the Stieltjes measure employed in its definition. We consider $t = 2, 0.5, 0$. More information about it can be found in [7].

The optimal solution for (1), applying this index is obtained solving the following boolean linear programming problem

$$\begin{aligned} \max \quad & cx + (cx - rx)/(t + 1) + \lambda(Rx - rx)/(t + 1) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{14}$$

3.3. *Fuzzy relations*

The method consists of the selection of the best alternative according to the concept of nondominated alternatives according to Orlovski [30]. A lot of fuzzy relations of ranking fuzzy numbers can be used. Among all of them, the two next will only be considered, because they are the most frequently employed.

3.3.1. *Degree of possibility of dominance of $\underset{\sim}{u}_i$ over $\underset{\sim}{u}_j$* [17]

Given two fuzzy numbers $\underset{\sim}{u}_i$, $\underset{\sim}{u}_j \in F(\mathbb{R})$, the degree of possibility is defined as

$$PD(\underset{\sim}{u}_i) = \operatorname{Poss}(\underset{\sim}{u}_i \geqslant \underset{\sim}{u}_j) = \operatorname*{Sup}_{z} \min\left(\mu_{u_i}(z), \operatorname*{Sup}_{v \leqslant z} \mu_{u_j}(v) \right)$$

$$= \operatorname*{Sup}_{\substack{z, v \\ z \geqslant v}} \min\left(\mu_{u_i}(z), \mu_{u_j}(v) \right).$$

For triangular fuzzy numbers $\underset{\sim}{c}x$ and $\underset{\sim}{c}y$ one has

$$R(\underset{\sim}{c}x, \underset{\sim}{c}y) = \operatorname{Poss}(\underset{\sim}{c}x \geqslant \underset{\sim}{c}y) = \begin{cases} 0 & \text{if } Rx \leqslant ry, \\ \dfrac{Rx - ry}{dx + d'y} & \text{if } cy > cx \text{ and } Rx > ry, \\ 1 & \text{if } cy \leqslant cx. \end{cases}$$

If a fuzzy strict preference relation is defined

$$\mu_S(\underset{\sim}{c}y, \underset{\sim}{c}x) = \max\{0, \mu_R(\underset{\sim}{c}y, \underset{\sim}{c}x) - \mu_R(\underset{\sim}{c}x, \underset{\sim}{c}y)\}$$

then,

$$\mu_S(\underset{\sim}{c}y, \underset{\sim}{c}x) = \begin{cases} 0 & \text{if } cy \leqslant cx, \\ \dfrac{cy - cx}{dx + d'y} & \text{if } cy > cx \text{ and } ry < Rx, \\ 1 & \text{if } ry \geqslant Rx \end{cases}$$

and the nondominated degree is defined as

$$\mu_{\mathrm{ND}}(\underset{\sim}{c}x) = 1 - \max_{y \in X} \mu_S(\underset{\sim}{c}y, \underset{\sim}{c}x).$$

Proposition 5. *Given the problem* (1), *$x^* \in X$ is the optimal solution for* (1) *using the degree of possibility of Dubois and Prade if x^* is the optimal solution for the following boolean linear programming problem*:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N. \end{aligned} \tag{15}$$

Proof. Let $x^* \in X$ such that $cx^* \geqslant cx \ \forall x \in X$, it is obvious that $\mu_S(\underset{\sim}{c}x, \underset{\sim}{c}x^*) = 0 \ \forall x \in X$ therefore $\mu_{\mathrm{ND}}(\underset{\sim}{c}x^*) = 1$, $\underset{\sim}{c}x^*$ has the nondominated degree 1 and $\underset{\sim}{c}x^*$ is nondominated in A. So that x^* is the optimal solution for (1), and it is the equivalent to obtaining the optimum for the following boolean linear programming problem:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0,1\}, \quad j \in N. \quad \square \end{aligned}$$

3.3.2. Degree of necessity of dominance of $\underset{\sim}{u}_i$ over $\underset{\sim}{u}_j$ [17]

$\underset{\sim}{u}_i$, $\underset{\sim}{u}_j \in F(\mathbb{R})$ being two fuzzy numbers, the degree of necessity is defined as

$$\mathrm{ND}(\underset{\sim}{u}_i) = \mathrm{Nec}(\underset{\sim}{u}_i \geqslant \underset{\sim}{u}_j) = \inf_{z} \operatorname*{Sup}_{\substack{v \\ v \leqslant z}} \max(1 - \mu_{u_i}(z), \mu_{u_j}(v)).$$

For triangular fuzzy numbers $\underset{\sim}{c}x$ and $\underset{\sim}{c}y$, it is reduced to

$$R(\underset{\sim}{c}x, \underset{\sim}{c}y) = \mathrm{Nec}(\underset{\sim}{c}x \geqslant \underset{\sim}{c}y) = \begin{cases} 0 & \text{if } cx \leqslant ry, \\ \dfrac{cx - ry}{d'x + d'y} & \text{if } cx > ry \text{ and } cy > rx, \\ 1 & \text{if } cy \leqslant rx \end{cases}$$

and the following strict preference relation is obtained

$$\mu_S(\underset{\sim}{c}y, \underset{\sim}{c}x) = \begin{cases} 0 & \text{if } cy \leqslant rx, \\ 0 \vee \dfrac{cy + ry - (cx + rx)}{d'x + d'y} & \text{if } cx > ry \text{ and } cy > rx, \\ 1 & \text{if } cx \leqslant ry \end{cases}$$

and the nondominated degree defined by

$$\mu_{\mathrm{ND}}(\underset{\sim}{c}x) = 1 - \max_{y \in X} \mu_S(\underset{\sim}{c}y, \underset{\sim}{c}x).$$

This nondominated degree can be obtained $\forall x' \in X$ in the following way: First, the problem

$$\begin{aligned} \max \quad & z(x') = \frac{cy + ry - (cx' + rx')}{d'(y + x')} \\ \text{s.t.} \quad & Ay \leqslant b, \\ & y_j \in \{0,1\}, \quad j \in N \end{aligned}$$

is solved for every $x' \in X$.

Next, the degree to which the element $\underset{\sim}{c}x'$ is dominated by none is

$$\mu_{\mathrm{ND}}(\underset{\sim}{c}x') = \begin{cases} 1 - z(x') & \text{if } z(x') \leqslant 1, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

the optimal solution for (1) is obtained as the points that belong to the set

$$\{x \in X / \mu_{\mathrm{ND}}(\underset{\sim}{c}x) \geqslant \mu_{\mathrm{ND}}(\underset{\sim}{c}y) \ \forall y \in X\}.$$

3.4. Linkage in the use of ranking functions and fuzzy relations

Kołodziejczyk studied conditions for the equivalence of the two aforementioned approaches, the use of ranking functions and the use of fuzzy relations [27]. The main result is the following:

"If R is the antireflexive and weakly transitive fuzzy relation described in $F(\mathbb{R}) \times F(\mathbb{R})$, and such that for any A, B, C, $D \in F(\mathbb{R})$,

$$\mu_R(A, B) > 0 \quad \text{and} \quad \mu_R(C, D) > 0 \;\Rightarrow\; \mu_R(A \oplus D, C \oplus B) > 0$$

then there is a ranking function f' such that if x^0 is the solution of (8) using f', then x^0 is a nondominated alternative with $\mu_{\mathrm{ND}}(\underset{\sim}{c}x^0) = 1$".

4. Approach using the representation theorem

Suppose the fuzzy costs taking part in the objective of (1), and consider $\forall c \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n)$,

$$\mu(c) = \operatorname*{Inf}_j \mu_j(c_j), \quad j \in N,$$

$\mu(\cdot)$ defines a fuzzy objective which induces a fuzzy preorder in X, as was shown in [42]. Consequently, using the representation theorem, a fuzzy solution to (1) can be found from the solution of the multiobjective parametric boolean linear programming problem

$$\max_{x \in X} \{cx / \forall c \in \mathbb{R}^n\colon \mu(c) \geqslant 1 - \alpha\}. \tag{17}$$

Taking into account that

$$\mu(c) \geqslant 1 - \alpha \;\Leftrightarrow\; \operatorname*{Inf}_j \mu_j(c_j) \geqslant 1 - \alpha \;\Leftrightarrow\; \mu_j(c_j) \geqslant 1 - \alpha, \quad j \in N, \; \alpha \in [0, 1]$$

from (4) it is obtained that

$$\mu_j(c_j) \geqslant 1 - \alpha \;\Leftrightarrow\; h_j^{-1}(1 - \alpha) \leqslant c_j \leqslant g_j^{-1}(1 - \alpha), \quad j \in N$$

and then denoting $\phi_j \equiv h_j^{-1}$, $\psi_j \equiv g_j^{-1}$, $j \in N$, problem (17) can be written as,

$$\max\{cx / x \in X,\; \Phi(1 - \alpha) \leqslant c \leqslant \Psi(1 - \alpha),\; \alpha \in [0, 1]\}, \tag{18}$$

where $\Phi(\cdot) = [\phi_1(\cdot), \ldots, \phi_n(\cdot)]$ and $\Psi = [\psi_1(\cdot), \ldots, \psi_n(\cdot)]$.

According to (4), with $h_j(\cdot)$ and $g_j(\cdot)$ linear functions, the interval $[\phi_j(1 - \alpha), \psi_j(1 - \alpha)]$ is $[c_j - \alpha(c_j - r_j), c_j + \alpha(R_j - c_j)]$, $j \in N$, $\alpha \in [0, 1]$, because

$$\phi_j(1 - \alpha) = h_j^{-1}(1 - \alpha) = c_j - \alpha(c_j - r_j), \quad \psi_j(1 - \alpha) \equiv g_j^{-1}(1 - \alpha) = c_j + \alpha(R_j - c_j), \qquad j \in N.$$

Moreover, if $\Gamma(1 - \alpha)$, $\alpha \in [0, 1]$, denotes the set of vectors $c \in \mathbb{R}^n$ with all of their components c_j being in the interval $[\phi_j(1 - \alpha), \psi_j(1 - \alpha)]$, $j \in N$, (18) can be finally written as,

$$\max\{cx / x \in X, c \in \Gamma(1 - \alpha), \alpha \in [0, 1]\}, \tag{19}$$

which for each $\alpha \in [0, 1]$ is a boolean interval multiobjective problem having in its objective functions costs that can assume values at respective intervals.

4.1. Solution methods

Now, two solution methods can be considered. The first by means of the criteria of efficient points and weight vector, and the second by means of the interval arithmetic.

4.1.1. Solution method with efficient points and weight vector

Suppose problem (1) and its auxiliary representation with interval objective function coefficients (19). Having fixed α, an interesting set of points of X to (19) is the set of efficient points. A point $x^* \in X$ is said to be efficient to (19) if there is no $x \in X$ such that $cx \geqslant cx^*$ $\forall c \in \Gamma(1-\alpha)$ with at least one strict inequality.

Let $S(1-\alpha)$ denote the set of those efficient points. Thus, in accordance with the representation theorem for fuzzy sets, it can be defined as

$$\underset{\sim}{S} = \bigcup_{\alpha} \alpha \cdot S(1-\alpha), \tag{20}$$

which is a fuzzy set giving the fuzzy solution to the former problem (1).

Clearly, for any $\alpha \in [0,1]$, $\Gamma(1-\alpha) \subset \mathbb{R}^n$ is a convex set with extreme points defined by $c \in \Gamma(1-\alpha)$ such that its components are in the intervals $[\phi_j(1-\alpha), \psi_j(1-\alpha)]$, $j \in N$, this characterization was studied in [3]. Following [3] as was developed in [15], obtaining the efficient points of (19) is equivalent to obtaining the efficient points for the problem

$$\begin{aligned} \max \quad & c_1 x_1 + \cdots + c_n x_n \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0,1\}, \quad j \in N, \\ & c \in E(1-\alpha), \quad \alpha \in [0,1] \end{aligned} \tag{21}$$

or more explicitly,

$$\begin{aligned} \max \quad & (c^1 x, c^2 x, \ldots, c^{2^n} x) \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0,1\}, \quad j \in N, \\ & c^k \in E(1-\alpha), \quad k = 1, \ldots, 2^n, \quad \alpha \in [0,1], \end{aligned} \tag{22}$$

which is a conventional parametric multiobjective boolean programming problem, and

$$\begin{aligned} E(1-\alpha) = \{ & [\phi_1(1-\alpha), \phi(1-\alpha), \ldots, \phi_n(1-\alpha)], [\psi_1(1-\alpha), \phi_2(1-\alpha), \ldots, \phi_n(1-\alpha)], \\ & \ldots, [\psi_1(1-\alpha), \ldots, \psi_{n-1}(1-\alpha), \phi_n(1-\alpha)], [\psi_1(1-\alpha), \ldots, \psi_{n-1}(1-\alpha), \psi_n(1-\alpha)]\}, \end{aligned}$$

where

$$\phi_j(0) = r_j, \quad \psi_j(0) = R_j, \quad \phi_j(1) = c_j, \quad \psi_j(1) = c_j \qquad \forall j \in N.$$

In order to obtain a fuzzy solution to (1) we rewrite (22) in a more convenient form. Let T be the index set, $T = \{1, \ldots, t\}$ with $t = 2^n$. We define the matrix $H(\alpha)$, $t \times n$, which contains the vectors of $E(1-\alpha)$ as rows. Then $H(\alpha) \cdot x$ is a vector of dimension $t \times 1$.

$$H(\alpha) \cdot x = (c^1 x, \ldots, c^t x), \quad c^k \in E(1-\alpha), \ k \in T.$$

Problem (22) can be expressed as

$$\begin{aligned}\max\quad & H(\alpha)\cdot x\\ \text{s.t.}\quad & Ax \leqslant b,\\ & x_j \in \{0,1\},\quad j\in N,\\ & h_{kj}(\alpha)\in\{\phi_j(1-\alpha),\psi_j(1-\alpha)\},\quad k\in T,\quad j\in N\\ & \alpha\in[0,1]\end{aligned} \tag{23}$$

denoted as $M(\alpha)$. We denote the set of efficient points of $M(\alpha)$ as $S(1-\alpha)$ because it coincides with the set of efficient points of (19).

Considering the partial order relation in $\mathbb{R}^n$, $\forall z, y\in\mathbb{R}^n$,

$$z \geqslant y \;\Leftrightarrow\; z_j \geqslant y_j,\quad j\in J=\{1,\ldots,n\},$$

$$z > y \;\Leftrightarrow\; z_j \geqslant y_j,\quad j\in J \text{ and } \exists i\in H z_i > y_j.$$

A point $x^*\in X$ is said to be efficient to $M(\alpha)$ if there is no $x\in X$ such that $H(\alpha)\cdot x > H(\alpha)\cdot x^*$.

Note that there exists an important difference between the set of efficient points of a multiobjective problem with real variables and a multiobjective boolean problem, because every efficient point in the real problem maximizes a linear functional of the type $\lambda\cdot H\cdot x$ for $\lambda\in\mathbb{R}^t\lambda > 0$ on the feasible set. In the latter case this may not happen. But we have the reciprocal as is shown in the following result.

Proposition 6. *Let $\beta\in\mathbb{R}^t$, $\beta > 0$ be such that x^* is an optimal solution to* $\max\{\beta\cdot H\cdot x/x\in X\}$ (P1) *then x^* is an efficient point of the problem* $\max\{H\cdot x/x\in X\}$ (P2).

Proof. Let x^* be an optimal solution to P1, and suppose that $x'\in X$ exists such that $H\cdot x' > H\cdot x^*$. Because $\beta > 0$ and $\beta\neq 0$ that implies $\beta\cdot H\cdot x' > \beta\cdot H\cdot x^*$ which is inconsistent with the optimality of x^*. Then x^* is an optimal solution (efficient point) to P2. □

In [41] a survey is given on the methods characterizing the set of efficient solutions of multiobjective discrete linear programming problems. These are too complex, with the added difficulty of the parameter α. This leads us to consider efficient solutions with an associated weight vector.

Definition 2. x^* is an efficient point for $M(\alpha)$ with weight vector w, $w\in\mathbb{R}^t$, $\sum w_k = 1$, $w_k\geqslant 0$ if x^* maximizes the following parametric boolean programming problem:

$$\begin{aligned}\max\quad & w\cdot H(\alpha)\cdot x\\ \text{s.t.}\quad & Ax \leqslant b,\\ & x_j \in \{0,1\},\quad j\in N,\\ & h_{kj}(\alpha)\in\{\phi_j(1-\alpha),\psi_j(1-\alpha)\},\quad k\in T,\quad j\in N,\\ & \alpha\in[0,1]\end{aligned} \tag{24}$$

denoted by $M_w(\alpha)$.

$M_w(\alpha)$ can be written as

$$\begin{aligned} \max \quad & (F + G\alpha)x \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N, \\ & \alpha \in [0, 1], \end{aligned} \tag{25}$$

where F, $G \in \mathbb{R}^n$ have the following expressions:

$$F = w \cdot H(0) = (c_1, \ldots, c_n)$$

and

$$G = w \cdot D,$$

with D a $t \times n$ matrix the rows of which are the vectors of the following set E',

$$E' = \{(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n), (\bar{b}_1, \underline{b}_2, \ldots, \underline{b}_n), \ldots, (\bar{b}_1, \ldots, \bar{b}_{n-1}, \underline{b}_n), (\bar{b}_1, \ldots, \bar{b}_n)\}$$

and $\underline{b}_j = -(c_j - r_j)$ and $\bar{b}_j = (R_j - c_j)$, that is, E' is the vector set, the components of which belong to $\{\underline{b}_j, \bar{b}_j\}$. Besides, these expressions of the vectors F and G are obtained in accordance with the expressions of $h_{ij}(\alpha)$,

$$h_{ij}(\alpha) \in \{\phi_j(1 - \alpha), \psi_j(1 - \alpha)\} = \{c_j - \alpha(c_j - r_j), c_j + \alpha(R_j - c_j)\}, \qquad j \in N, \quad \alpha \in [0, 1].$$

$S_w(1 - \alpha)$ denoting the set of optimal solutions to $M_w(\alpha)$ and according to the representation theorem, we can define the fuzzy solution with weight w to (1) as

$$\underset{\sim}{S}_w = \bigcup_{\alpha} \alpha \cdot S_w(1 - \alpha),$$

which is a subset of the fuzzy solution (20), because $S_w(1 - \alpha) \subset S(1 - \alpha)$.

With regard to the weight vector $w \in \mathbb{R}^t$, its choice depends on the concrete situations and the preference of the decision-maker about the extremes to weigh up.

Finally, we should point out that in [26] there is a list of methods to solve parametric discrete programming problems such as (25).

4.1.2. Solution method with interval arithmetic

We suppose the auxiliary boolean interval multiobjective problem, (19),

$$\max\{cx/x \in X, \; c \in \Gamma(1 - \alpha), \; \alpha \in [0, 1]\}.$$

In [24] Ishibuchi and Tanaka use order relations which represent the decision makers' preference between interval costs. To maximize the interval objective function, the order relations are defined by the right limit, the left limit, the center and the width of an interval. The maximization problem with the interval objective function is converted into a biobjective problem using the order relations.

According to these results, (19) can be rewritten using the above criteria as the following parametric biobjective programming problem,

$$\max\{z' = (z^l(x, \alpha), z^c(x, \alpha))/x \in X \subset \mathbb{R}^n, \; \alpha \in (0, 1]\}, \tag{26}$$

denoted by $N(\alpha)$, where $z^l(x, \alpha)$ and $z^c(x, \alpha)$ are defined by

$$z^l(x, \alpha) = \sum_{j=1}^{n} [c_j - \alpha(c_j - r_j)]x_j,$$

$$z^c(x, \alpha) = (1/2) \sum_{j=1}^{n} [2c_j + \alpha[R_j + r_j - 2c_j)]x_j.$$

A fuzzy solution for (1) can be found from the parametric solution of the biobjective parametric problem. If we define by $S_N(1 - \alpha)$ the set of efficient solutions of (26), then in accordance with the representation theorem for fuzzy sets one can define

$$\underset{\sim}{S}_N = \bigcup_{\alpha} \alpha \cdot S(1 - \alpha), \tag{27}$$

which is a fuzzy set giving the fuzzy solution to the former problem, using interval arithmetic.

Concretely, if we consider a weight vector $\beta \in \mathbb{R}^2$, $\beta_i \in [0, 1]$ to each of the objectives taking part in (26), in such a way that $\beta_1 + \beta_2 = 1$, then a solution to (1) can be found from the solution of the problem, $N_\beta(\alpha)$,

$$\max\{(\beta_1 \cdot z^l(x, \alpha) + \beta_2 \cdot z^c(x, \alpha))/x \in X \subset \mathbb{R}^n,\ \alpha \in (0, 1]\}. \tag{28}$$

As is evident, (28) is a conventional parametric boolean linear programming problem. If the set of optimal points for (28) is defined as $S_\beta(1 - \alpha)$ for every $\alpha \in (0, 1]$, then the fuzzy solution with weight β will be given by the fuzzy set

$$\underset{\sim}{S}_\beta = \bigcup_{\alpha} \alpha \cdot S_\beta(1 - \alpha), \tag{29}$$

which is a subset of the fuzzy solution (27).

4.2. *Relating the two solution methods*

Proposition 7. *The problem $N_\beta(\alpha)$, (28), is a specific problem of $M_w(\alpha)$, (24).*

Proof. It is enough to assign the weight vector $w \in \mathbb{R}^t$ in the following form: $w_1 = (\beta_1 + 1/2\beta_2)$, $w_t = 1/2\beta_2$ and $w_i = 0$, $2 \leqslant i < t$. □

So, the problem $N_\beta(\alpha)$ can be expressed as

$$\begin{aligned} \max \quad & (F + G\alpha)x \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x_j \in \{0, 1\}, \quad j \in N, \\ & \alpha \in [0, 1], \end{aligned} \tag{30}$$

where $F, G \in \mathbb{R}^n$,

$$F = (c_1, \ldots, c_n)$$

and

$$G = (\beta_1 + 1/2\beta_2) \cdot (\underline{b}_1, \ldots, \underline{b}_n) + 1/2\beta_2(\bar{b}_1, \ldots, \bar{b}_n),$$

with $\underline{b}_j = -(c_j - r_j)$ and $\bar{b}_j = (R_j - c_j)$.

5. Numerical example

We consider the example presented at the beginning of this paper, which was formulated as follows:

$$\begin{aligned} \max \quad & \underset{\sim}{55}x_1 + \underset{\sim}{40}x_2 + \underset{\sim}{50}x_3 + \underset{\sim}{28}x_4 + \underset{\sim}{35}x_5 + \underset{\sim}{43}x_6 \\ \text{s.t.} \quad & 14x_1 + 11x_2 + 17x_3 + 7x_4 + 13x_5 + 10x_6 \leqslant 32, \\ & x_1 + x_2 + x_3 \geqslant 1, \\ & x_3 + x_5 + x_6 \geqslant 1, \\ & x_j \in \{0, 1\}, \quad 1 \leqslant j \leqslant 6, \end{aligned}$$

where the fuzzy costs are the following triangular fuzzy numbers:

$$\underset{\sim}{c}_1 = (47, 55, 63), \qquad \underset{\sim}{c}_2 = (35, 40, 45), \qquad \underset{\sim}{c}_3 = (38, 50, 62),$$
$$\underset{\sim}{c}_4 = (18, 28, 38), \qquad \underset{\sim}{c}_5 = (28, 35, 42), \qquad \underset{\sim}{c}_6 = (31, 43, 55).$$

(i) Solutions using some ranking fuzzy number methods
Method of Jain $z_{\max} = 156.00$ $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x^*) = (96,\ 126,\ 156)$.
Method of Chang $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x^*) = (96,\ 126,\ 156)$.
First method of Yager $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x^*) = (96,\ 126,\ 156)$.
Second method of Yager $x^* = (0\ 1\ 0\ 0\ 1\ 0)$, $g(x^*) = (63,\ 75,\ 87)$.
Third method of Yager $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x^*) = (96,\ 126,\ 156)$.
Method of Adamo $\forall \alpha \in [0, 1]$ $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x(\alpha)) = (96,\ 126,\ 156)$.
Average value $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x^*) = (96,\ 126,\ 156)$.
With the same solution for: $\lambda = 1, 0.5, 0$ and $t = 2, 1, 0.5$.
Degree of possibility $x^* = (1\ 0\ 0\ 1\ 0\ 1)$, $g(x^*) = (96,\ 126,\ 156)$.
Degree of necessity $x^* = (0\ 1\ 0\ 1\ 0\ 1)$, $g(x^*) = (84,\ 111,\ 138)$.
With degree of nondominance: 1.000.
(ii) Solutions using the representation theorem.
The extreme functions using the representation theorem are:

$$\phi_1(1-\alpha) = 55 - 8\alpha, \quad \psi_1(1-\alpha) = 55 + 8\alpha, \quad \phi_2(1-\alpha) = 40 - 5\alpha, \quad \psi_2(1-\alpha) = 40 + 5\alpha,$$
$$\phi_3(1-\alpha) = 50 - 12\alpha, \quad \psi_3(1-\alpha) = 50 + 12\alpha, \quad \phi_4(1-\alpha) = 28 - 10\alpha, \quad \psi_4(1-\alpha) = 28 + 10\alpha,$$
$$\phi_5(1-\alpha) = 35 - 7\alpha, \quad \psi_5(1-\alpha) = 35 + 7\alpha, \quad \phi_6(1-\alpha) = 43 - 12\alpha, \quad \psi_6(1-\alpha) = 43 + 12\alpha.$$

Weight vectors

We consider the next two vectors, w^1, $w_1^1 = 1$, $w_i^1 = 0$, $2 \leqslant i \leqslant 16$, and w^2, $w_i^2 = 0$, $1 \leqslant i \leqslant 15$, $w_{16}^2 = 1$. These can be considered as the most pessimistic and the most optimistic vector values, respectively.

The problem $M_{w^1}(\alpha)$ is

$$\begin{aligned} \max \quad & (55 - 8\alpha)x_1 + (40 - 5\alpha)x_2 + (50 - 12\alpha)x_3 + (28 - 10\alpha)x_4 + (35 - 7\alpha)x_5 + (43 - 12\alpha)x_6 \\ \text{s.t.} \quad & 14x_1 + 11x_2 + 17x_3 + 7x_4 + 13x_5 + 10x_6 \leqslant 32, \\ & x_1 + x_2 + x_3 \geqslant 1, \\ & x_3 + x_5 + x_6 \geqslant 1, \\ & x_j \in \{0, 1\}, \quad 1 \leqslant j \leqslant 6, \quad \alpha \in [0, 1] \end{aligned}$$

and its solution is

$$x(\alpha) = (1, 0, 0, 1, 0, 1) \quad \forall \alpha \in [0, 1],$$
$$z(\alpha) = 126 - 30\alpha \quad \forall \alpha \in [0, 1],$$
$$\underset{\sim}{S}_{w^1} = \{(1, 0, 0, 1, 0, 1)/1\}.$$

The problem $M_{w^2}(\alpha)$ is the following:

$$\begin{aligned}
\max \quad & (55 + 8\alpha)x_1 + (40 + 5\alpha)x_2 + (50 + 12\alpha)x_3 + (28 + 10\alpha)x_4 + (35 + 7\alpha)x_5 + (43 + 12\alpha)x_6 \\
\text{s.t.} \quad & 14x_1 + 11x_2 + 17x_3 + 7x_4 + 13x_5 + 10x_6 \leqslant 32, \\
& \quad x_1 + x_2 + x_3 \geqslant 1, \\
& \quad x_3 + x_5 + x_6 \geqslant 1, \\
& \quad x_j \in \{0, 1\}, \quad 1 \leqslant j \leqslant 6, \quad \alpha \in [0, 1],
\end{aligned}$$

the optimal solution of which coincides with the solution of $M(\alpha)$

$$x(\alpha) = (1, 0, 0, 1, 0, 1) \quad \forall \alpha \in [0, 1],$$

$$z(\alpha) = 126 + 30\alpha \quad \forall \alpha \in [0, 1],$$

$$\underset{\sim}{S}_{w^2} = \{(1, 0, 0, 1, 0, 1)/1\}.$$

Interval arithmetic

$$z^c(x, \alpha) = 55x_1 + 40x_2 + 50x_3 + 28x_4 + 35x_5 + 43x_6,$$

$$z^l(x, \alpha) = (55 - 8\alpha)x_1 + (40 - 5\alpha)x_2 + (50 - 12\alpha)x_3 + (28 - 10\alpha)x_4 + (35 - 7\alpha)x_5 + (43 - 12\alpha)x_6$$

and solving the biobjective problem for whatever weight vector $\beta \in \mathbb{R}^2$, we obtain the same solution as that obtained above.

Fig. 1. represents the fuzzy costs associated to the different solutions for this example.

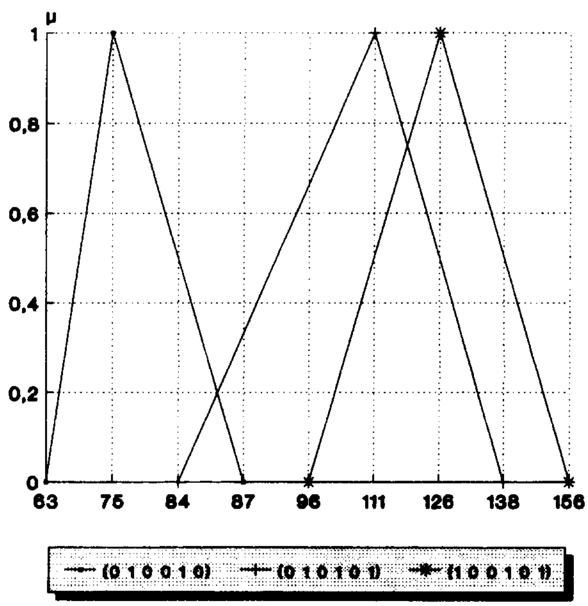


Fig. 1. FBLP with fuzzy costs.

6. Final remarks

We have studied in greater depth the different methods to solve the FBLP problems with fuzzy costs. The use of methods to rank fuzzy numbers lead us to obtain classical boolean programming problems as auxiliary models whereas the use of the representation theorem leads us to obtain parametric interval boolean programming problems. Different ways to solve the parametric interval programming problems have been proposed.

Other different approaches to solve the fuzzy linear programming problem with fuzzy costs have been presented in [31, 40, 37, 12, 29]. In [16] it was shown how to obtain some of them [16, 37, 40] as particular models of the parametric multiobjective problems. The rest of the approaches use the concept of α-level and therefore the solution methods obtain efficient solutions of the interval multiobjective problem (19), when they are used for solving the former problem.

Finally, we can conclude that, according to the preference of the decision maker and the nature of the problem, different solutions methods may be used to solve the former problem and different solutions can be obtained as we can see in the example. The possibility of obtaining different solutions is in accordance with the imprecise raising of the problem.

Future research is to design a general interactive decision support system for fuzzy boolean programming problems, with the possibility for the users to select the different solution approaches in a friendly environment, and to choose the final solution according to their preferences. Studies in this direction for fuzzy linear programming have been carried out (see [28, 30, 6]).

References

[1] Z.A. Azmi, An approach for classical and fuzzy scheduling, in: J.L. Verdegay, M. Delgado, Eds., *The Interface between Artificial Intelligence and Operations Research in Fuzzy Environment* (TÜV Rheinland, Köln, 1989) 1–11.

[2] R.E. Bellman and L.A. Zadeh, Decision making in a fuzzy environment, *Management Sci.* **17**(4) (1970) 141–164.

[3] G.R. Bitran, Linear multiple objective problems with interval coefficients, *Management Sci.* **26** (1980) 694–706.

[4] G. Bortolan and R. Degani, A review of some methods for ranking fuzzy subsets, *Fuzzy Sets and Systems* **15** (1985) 1–19.

[5] D.E. Brown and C.C. White, *Operations Research and Artificial Intelligence: The Integration of Problem-Solving Strategies*, Part III (Kluwer Academic Publishers, London, 1990).

[6] J.M. Cadenas, Design and implementation of an interactive system for solving fuzzy optimization problems, Ph.D., Dissertation, University of Granada, 1993.

[7] L. Campos and A. Gonzalez, Further contributions to the study of the average value for ranking fuzzy numbers, *Internat. J. Approx. Reason.* **10** (1994) 135–153.

[8] J.L. Castro and F. Herrera, Approaching fuzzy boolean linear programming problems with imprecise costs, *Fourth IFSA Congress*, Brussels, 1991, Computer, Management and Systems Science, 31–34.

[9] J.L. Castro, F. Herrera and J.L. Verdegay, Solving Linear Boolean Programming Problems with Imprecise Costs, *Proc. of IEEE Internat. Conf. on Fuzzy Systems* (San Diego, 1992) 1025–1032.

[10] J.L. Castro, F. Herrera and J.L. Verdegay, Knowledge based systems and fuzzy boolean programming, *Internat. J. Intelligent Systems* **9** (1994) 211–225.

[11] T.M. Cavalier, P.M. Pardalos and A.L. Soyster, Modeling and integer programming techniques applied to propositional calculus, *Comput. Oper. Res.* **17** (1990) 561–570.

[12] S. Chanas and D. Kuchta, Linear programming problem with fuzzy coefficients in the objective function, Technical University of Wroclaw, Report Serii PRE Nr. 38, Wroclaw, 1992.

[13] W. Chang, Ranking of fuzzy utilities with triangular membership functions, *Proc. Internat. Conf. on Policy Anal. and Inf. Systems* (1981) 263–272.

[14] J. Darzentas, On fuzzy location models, in: J. Kacprzyk and S.A. Orlovski, Eds., *Optimization Models using Fuzzy Sets and Possibility Theory* (D. Reidel, Boston, 1987) 327–241.

[15] M. Delgado, J.L. Verdegay and M.A. Vila, Imprecise costs in mathematical programming problems, *Control Cybernet.* **16** (1987) 113–121.

[16] M. Delgado, J.L. Verdegay and M.A. Vila, Relating different approaches to solve linear programming problems with imprecise costs, *Fuzzy Sets and Systems* **37** (1990) 32–42.

[17] D. Dubois and H. Prade, Ranking of fuzzy numbers in the setting of possibility theory, *Inform. Sci.* **30** (1983) 183–244.
[18] C. Fabian and M. Stoica, Fuzzy integer programming, in: H.J. Zimmermann, L.A. Zadeh and B.R. Gaines, Eds., *Fuzzy Sets and Decision Analysis* (North-Holland, Amsterdam, 1984) 133–145.
[19] M. Fedrizzi, J. Kacprzyk and J.L. Verdegay, A survey of fuzzy optimization and mathematical programming, in: M. Fedrizzi, J. Kacprzyk and M. Roubens, Eds., *Interactive Fuzzy Optimization* (Springer, Berlin, 1991) 15–28.
[20] F. Glover, Future paths for integer programming and links to artificial intelligence, *Comput. Oper. Res.* **13** (1986) 533–549.
[21] F. Herrera and J.L. Verdegay, Approaching fuzzy integer linear programming problems, in: M. Fedrizzi, J. Kacprzyk and M. Roubens, eds., *Interactive Fuzzy Optimization* (Springer, Berlin, 1991) 78–91.
[22] F. Herrera, J.L. Verdegay and H.J. Zimmermann, Boolean programming problems with fuzzy constraints, *Fuzzy Sets and Systems* **56** (1993) 285–293.
[23] H. Ishibuchi and H. Tanaka, Multiobjective programming in optimization of the interval objective function, *European J. Oper. Res.* **48** (1990) 219–225.
[24] R. Jain, A procedure for multiple-aspect decision-making using fuzzy sets, *Internat. J. Systems Sci.* **8** (1977) 1–7.
[25] B. Jaumard, P.S. Ow and B. Simeone, A selected artificial intelligence bibliography for operations researchers, *Ann. Oper. Res.* **12** (1988) 1–50.
[26] L. Jenkins, Parametric methods in integer linear programming, *Ann. Oper. Res.* **27** (1990) 77–96.
[27] W. Kolodziejczyk, On equivalence of two optimization methods for fuzzy discrete programming problems, *European J. Oper. Res.* **36** (1988) 85–91.
[28] Y.-J. Lai and C.-L. Hwang, Interactive fuzzy linear programming, *Fuzzy Sets and Systems* **45** (1992) 169–183.
[29] Y.-J. Lai and C.-L. Hwang, Possibilistic linear programming for managing interest rate risk, *Fuzzy Sets and Systems* **54** (1993) 135–146.
[30] Y.-J. Lai and C.-L. Hwang, IFLP-II: A decision support system, *Fuzzy Sets and Systems* **54** (1993) 47–56.
[31] M.K. Luhandjula, Linear programming with a possibilistic objective function, *European J. Oper. Res.* **31** (1987) 110–117.
[32] S.A. Orlovski, Decision-making with a fuzzy preference relation, *Fuzzy Sets and Systems* **1** (1978) 155–167.
[33] S.A. Orlovski, Mathematical programming problems with fuzzy parameters, in: J. Kacprzyk and R.R. Yager, Eds., *Management Decision Support Systems Using Fuzzy Sets and Possibility Theory* (Rheinland, New York, 1985) 136–145.
[34] H. Rommelfanger, R. Hanuscheck and J. Wolf, Linear programming with fuzzy objectives, *Fuzzy Sets and Systems* **29** (1989) 31–48.
[35] M. Sakawa and T. Mitani, Fuzzy 0–1 programming through neural computation, *Proc. of IFSA'91-CMSS* (1991) 217–220.
[36] H.M. Salkin and K. Mathur, *Foundations of Integer Programming* (North-Holland, New York, 1989).
[37] R.E. Steuer, Algorithms for linear programming problems with interval objective function coefficients, *Math. Oper. Res.* **6** (1981) 333–348.
[38] H.A. Taha, *Integer Programming. Theory, Applications and Computations* (Academic Press, New York, 1975).
[39] M. Tada and H. Ishii, Some fuzzy versions of scheduling problems, *Proc. of IFSA'91-CMSS* (1991) 268–271.
[40] H. Tanaka, H. Ichihashi and K. Asai, A formulation of fuzzy linear programming problems based on comparison of fuzzy numbers, *Control Cybernet.* **3** (1984) 185–194.
[41] J. Teghem Jr. and P.L. Kunsch, A survey of techniques for finding efficient solutions to multi-objective integer linear programming, *Asia-Pacific J. Oper. Res.* **3** (1986) 95–108.
[42] J.L. Verdegay, A dual approach to solve the fuzzy linear programming problem, *Fuzzy Sets and Systems* **14** (1984) 131–141.
[43] H.P. Williams, Linear and integer programming applied to the propositional calculus, *Systems Res. Inform. Sci.* **2** (1987) 81–100.
[44] J.M. Wilson, Compact normal forms in propositional logic and integer programming formulations, *Comput. Oper. Res.* **17** (1990) 309–314.
[45] R.R. Yager, Ranking fuzzy subsets over the unit interval. *Proc. 1978 CDC* (1978) 1435–1437.
[46] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, *Inform. Sci.* **24** (1981) 143–161.
[47] R.R. Yager, A mathematical programming approach to inference with the capability of implementing default rules, *Internat. Man-Mach. Stud.* **29** (1988) 685–714.
[48] H.J. Zimmermann and M.A. Pollatschek, Fuzzy 0–1 linear programs, in: H.J. Zimmermann, L.A. Zadeh and B.R. Gaines, Eds., *Fuzzy Sets and Decision Analysis* (North-Holland, Amsterdam, 1984) 133–145.
[49] K. Zimmermann, Fuzzy sets covering problems, *Internat. J. Gen. Systems* **20** (1991) 127–131.
[50] Q. Zhu and E.S. Lee, Comparison and ranking of fuzzy numbers, in: Kacprzyk and M. Fedrizzi, Eds., Fuzzy Regression Analysis (Omnitech Press and Physica-Verlag, Wurzburg, 1992) 21–44.