Fuzzy Sets and Systems 55 (1993) 285–293
North-Holland

Boolean programming problems with fuzzy constraints*

F. Herrera and J.L. Verdegay
Department of Computer Science and Artificial Intelligence, University of Granada, 18071-Granada, Spain

H.-J. Zimmermann
Lehrstuhl für Unternehmensforschung, RWTH Aachen, Templergraben 64, W-5100 Aachen, Germany

Received June 1992
Revised September 1992

Abstract: Boolean linear programming problems for which there exist some lack of precision of vague nature in the constraints are considered. An α-cut-based approach is considered to solve them. Then, an algorithm providing a fuzzy solution is proposed and analyzed. The linking with other known solution methods is also studied.

Keywords: Fuzzy boolean programming; fuzzy constraints; mathematical algorithm; t-norms; linkage.

1. Introduction

Boolean Linear Programming (BLP) problems have a lot of relevant applications in many fields, as for instance those related to Artificial Intelligence and Operations Research. In particular, they are specially important for representing and reasoning with propositional knowledge [2, 9].

As it occurs in the conventional case of Linear Programming problems, some lack of precision of vague nature may be assumed in the formulation of BPL problems. In this case, Fuzzy Boolean Linear Programming (FBLP) problems can be considered. This kind of models were formerly introduced and studied in [12]. Also in [3] an extensive classification of them and a description of each of the possible problems can be found.

Considering a general BLP problem with imprecisely defined (fuzzy) constraints, i.e., a FBLP problem, similarly to the conventional case, this problem can be written as

$$\max\ z = cx \tag{1a}$$

$$\text{s.t.}\ \sum_{j\in N} a_{ij}x_j \lesssim b_i,\quad i\in M=\{1,\ldots,m\}, \tag{1b}$$

$$x_j\in\{0,1\},\quad j\in N=\{1,\ldots,n\} \tag{1c}$$

where as usual $c\in\mathbf{R}^n$, a_{ij}, $b_i\in\mathbf{R}$, $i\in M$, $j\in N$, and the symbol $\lesssim$ means the decision-maker is willing to permit some violations of the constraints, that is, he assumes fuzzy constraints characterized by membership functions

$$\mu_i:\mathbf{R}^n \to (0,1],\quad i\in M, \tag{2}$$

each of them giving the degree to which each $x\in\mathbf{R}^n$ accomplishes the ith constraint.

Correspondence to: Mr. J.L. Verdegay, Department of Computer Science and Artificial Intelligence, University of Granada, 18071-Granada, Spain.

* This research has been partially supported by DGICYT PS88-0090.

In [3] and [12] two different solution methods for (1) were proposed. Hence, the main aim of this paper is to relate both methods. To do this, the paper is set up as follows. In the next section the two different approaches solving the problem are introduced. Section 3 shows the link between both methods and their respective solutions. Finally a numerical example clarifying the above relationships is analyzed and some remarks are added.

2. Solving Boolean programming models with fuzzy constraints

The starting point is the following FBLP model:

$$\begin{aligned} &\min \ z = f(x) && \text{(3a)}\\ &\text{s.t.} \ \ g_i(x) \lesssim b_i, \quad i \in M, && \text{(3b)}\\ &\qquad x_j \in \{0, 1\}, \quad j \in N. && \text{(3c)} \end{aligned}$$

In this paper $f(x)$ and each $g_i(\cdot)$, $\forall i \in M$, are assumed to be linear functions.

To solve (3) two approaches can be considered. The first one, [12], looks for a solution according to the classical Bellman and Zadeh's concept of maximizing decision [1]. The second one [3] seeks a solution to be obtained from the Representation Theorem for Fuzzy Sets [5], that is, by taking advantage of the α-cuts of the fuzzy constraint set. In the following both ways are described.

2.1. First approach

The problem considered in [12] is

$$\begin{aligned} &\text{Find} \ \ x_j \in \{0, 1\}, \quad j \in N, && \text{(4a)}\\ &\text{such that} \ \ \sum_{j \in N} a_{ij} x_j \lesssim b_i, \quad i \in M, && \text{(4b)} \end{aligned}$$

where no distinction is made between fuzzy objective and fuzzy constraints, and the linear membership functions are defined as

$$\mu_i(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b_i, \\ 1 - (a_i x - b_i)/d_i & \text{if } b_i \leqslant a_i x \leqslant b_i + d_i, \\ 0 & \text{if } a_i x \geqslant b_i + d_i. \end{cases}$$

Note that (4) is a special case of (3), in which a fuzzy goal is assumed for the values of the objective function. If the objective function calls for strict maximization (or minimization) it can also be transformed into a fuzzy set (see Zimmermann [14, p. 256]). Even for nonlinear membership functions similar results may be achieved (see Zimmermann [13, pp. 101–106]).

From Bellman and Zadeh's concept of maximizing decision, the solution of (4) will be x^*, satisfying

$$\mu_D(x^*) = \max_x \mu_D(x) = \max_x \min_i [\mu_1(x), \ldots, \mu_m(x)]. \tag{5}$$

Then, to obtain x^*, the following mixed BLP problem is considered:

$$\begin{aligned} &\max \ \lambda && \text{(6a)}\\ &\text{s.t.} \ \ \lambda \leqslant 1 - (a_i x - b_i)/d_i, \quad i \in M, && \text{(6b)}\\ &\qquad x_j \in \{0, 1\}, \quad j \in N, && \text{(6c)}\\ &\qquad 0 \leqslant \lambda \leqslant 1. && \text{(6d)} \end{aligned}$$

If (x^*, λ^*) is obviously the optimal solution of (6), its x-part (x^*) is also the maximizing decision in (5).

In [12], for computational reasons, (6) is transformed into the equivalent problem

$$\max \quad \left\{0, \min\left[1, 1 + \max_x \min_i\left(\bar{b}_i - \sum_{j\in N} \overline{a_{ij}}\, x_j\right)\right]\right\}, \quad x_j \in \{0, 1\}, \quad j \in N, \tag{7}$$

where $\bar{b}_i = b_i/d_i$ and $\overline{a_{ij}} = a_{ij}/d_i$.

Thus, for

$$\bar{\lambda} = \max_x \min_i\left(b_i - \sum_{j\in N} \overline{a_{ij}}\, x_j\right) \tag{8}$$

the problem which eventually has to be solved is

$$\text{Max } \{0, \min[1 + \bar{\lambda}]\} \tag{9a}$$

$$\text{s.t.} \quad x_j \in \{0, 1\}, \quad j \in N. \tag{9b}$$

2.2. Second approach

Consider the FBLP problem (3), and let

$$\mu_i(x) = \begin{cases} 1 & \text{if } g_i(x) \leqslant b_i, \\ f_i(g_i(x)) & \text{if } b_i \leqslant g_i(x) \leqslant b_i + d_i, \\ 0 & \text{if } g_i(x) \geqslant b_i + d, \end{cases}$$

be the membership function of the ith constraint, $i \in M$.

Denote for each constraint

$$X_i = \{x \in \mathbf{R}^n \mid g_i(x) \leqslant b_i, x_j \in \{0, 1\}\}, \quad i \in M.$$

If $X = \bigcap_{i\in M} X_i$, then (3) can be rewritten as

$$\min \quad \{z = f(x)/x \in X\}. \tag{10}$$

Any α-cut, $\forall\alpha \in (0, 1]$, of the constraint set is a classical set $X(\alpha) = \{x \in \mathbf{R}^n \mid \mu_X(x) \geqslant \alpha\}$ where $\forall x \in \mathbf{R}^n$, $\mu_X(x) = \inf\{\mu_i(x), i \in M\}$. Thus, $X_i(\alpha)$ denotes on α-cut of the ith constraint, $i \in M$.

As, $\forall\alpha \in (0, 1]$,

$$X(\alpha) = \bigcap_{i\in M} \{x \in \mathbf{R}^n \mid g_i(x) \leqslant r_i(\alpha), x_j \in \{0, 1\}\} \quad \text{with } r_i(\alpha) = f_i^{-1}(\alpha),$$

(3) can be written as

$$\min \quad z = f(x) \tag{11a}$$

$$\text{s.t.} \quad g_i(x) \leqslant r_i(\alpha), \quad i \in M \tag{11b}$$

$$x_j \in \{0, 1\}, \quad \alpha \in (0, 1], \quad j \in N. \tag{11c}$$

Then denoting $\forall\alpha \in (0, 1]$ by $S(\alpha) = \{x \in \mathbf{R}^n \mid f(x) = \min f(y), y \in X(\alpha)\}$, a fuzzy solution concept for (3) based on the corresponding one by Orlovski [6] can be defined.

Definition. Given a FBLP problem like (3), the fuzzy set defined by the membership function

$$\lambda(x) = \begin{cases} \sup_{x\in S(\alpha)} \alpha, & x \in \bigcup_\alpha S(\alpha), \\ 0 & \text{elsewhere,} \end{cases}$$

is the fuzzy solution of the problem.

$\lambda(\cdot)$ gives the degree of goods alternatives belonging to the fuzzy solution, and the decision-maker will make the final choice himself.

Consider, in particular, the following FBLP problem in which, without loss of generality, one supposes that all the coefficients are integer numbers:

$$\min \ z = cx \tag{12a}$$

$$\text{s.t.} \quad a_i x \lesssim b_i, \quad i \in M, \tag{12b}$$

$$x_j \in \{0, 1\}, \quad j \in N. \tag{12c}$$

The membership functions of the constraints are defined by

$$\mu_i(x) = \begin{cases} 1 & \text{if } a_i x \leq b_i, \\ [(b_i + d_i) - a_i x]/d_i & \text{if } b_i \leq a_i x \leq b_i + d_i, \\ 0 & \text{if } a_i x \geq b_i + d_i. \end{cases}$$

Thus, (12) can be written as the following conventional parametric BLP problem:

$$\min \ z = cx \tag{13a}$$

$$\text{s.t.} \quad a_i x \leq bi + d_i(1 - \alpha), \quad i \in M, \tag{13b}$$

$$x_j \in \{0, 1\}, \quad \alpha \in (0, 1], \quad j \in N. \tag{13c}$$

In the following, to approach a solution method for (13) which provides the FLBP problem (12) with a fuzzy solution, for each fixed $\alpha \in (0, 1]$, (13) will be denoted by $\mathrm{P}(\alpha)$ and will be used as an intermediate problem. The optimal solution of $\mathrm{P}(\alpha)$ will be denoted by $x(\alpha)$ and, as usually, $[x]$ will represent the largest integer less than or equal to some value x. Thus, if $\{x\}$ denotes the fractional remainder of $x \in \mathbf{R}$, then $x = [x] + \{x\}$, $0 \leq \{x\} < 1$.

Proposition 1. *Let $\alpha' \in (0, 1]$ be a specific fixed value and $x(\alpha')$ the optimal 0–1 solution of the corresponding problem $\mathrm{P}(\alpha)$. Then*

$$\lambda(x(\alpha')) = \min\{\mu_i(x(\alpha')), i \in M\}.$$

Proof. Let $\theta = \min\{\mu_i(x(\alpha')), i \in M\}$. Then $\theta \geq \alpha'$ is satisfied and $X(\theta) \subseteq X(\alpha')$. As $c \cdot x(\alpha') = \min\{c \cdot x \mid x \in X(\alpha')\}$ and $x(\alpha') \in X(\theta) \subseteq X(\alpha')$, it follows that $c \cdot x(\alpha') = \min\{c \cdot x \mid x \in X(\theta)\}$. Finally, as $\forall \alpha > \theta$, $x(\alpha') \notin X(\alpha)$, then it follows that $\lambda(x(\alpha')) = \theta = \min\{\mu_i(x(\alpha')), i \in M\}$. □

Corollary. *Let $x(\alpha')$ be an optima 0–1 solution of $\mathrm{P}(\alpha')$ for some fixed $\alpha' \in (0, 1]$, and $\theta = \lambda(x(\alpha'))$. Then, $\forall \alpha \in [\alpha', \theta]$, $x(\alpha')$ is also an optimal solution of $\mathrm{P}(\alpha)$.*

Proof. It is clear that $\forall \alpha \in [\alpha', \theta]$, $X(\theta) \subseteq X(\alpha) \subseteq X(\alpha')$. Therefore, as $x(\alpha') \in X(\theta)$ and $c \cdot x(\alpha') = \min\{c \cdot x \mid x \in X(\alpha')\}$, it follows that $c \cdot x(\alpha') = \min\{c \cdot x \mid x \in X(\alpha)\}$. □

Remark 1. Notice that from the above it follows that $\forall \alpha \in [\alpha', \theta]$, $x(\alpha') = x(\alpha)$ and, in particular, if $\alpha' \neq \alpha$, then $\lambda(x(\alpha')) = \lambda(x(\alpha)) = \theta$.

Remark 2. The following relation will be needed. Consider, $\forall \alpha \in (0, 1]$,

$$1_\alpha = \begin{cases} 1 & \text{if } \max\limits_i\{b_i + d_i(1 - \alpha)\} = 0, \\ \min\limits_i\{\{b_i + d_i(1 - \alpha)\}: \{b_i + d_i(1 - \alpha)\} \neq 0\} & \text{elsewhere.} \end{cases}$$

If $d' = \max\{d_i\}$ and $\Delta\alpha = 1_\alpha / d'$, then it is evident that $d_i \cdot \Delta\alpha \leq 1, \quad \forall i \in M$.

Lemma. *If* $\{b_i + d_i(1-\alpha)\} \neq 0$, *then* $[b_i + d_i(1-\alpha)] = [b_i + d_i(1-(\alpha+\Delta\alpha))]$.

Proof. In fact, using the above notation, it is clear that

$$\begin{aligned} 1 &> \{b_i + d_i(1-\alpha)\} - d_i \cdot \Delta\alpha = \{b_i + d_i(1-\alpha)\} - d_i \cdot l_\alpha / d' \\ &\geqslant \{b_i + d_i(1-\alpha)\} - d_i \cdot l_\alpha / d_i = \{b_i + d_i(1-\alpha)\} - 1_\alpha \geqslant 0, \end{aligned}$$

and therefore

$$[b_i + d_i(1-\alpha)] = [b_i + d_i(1-\alpha) - d_i \cdot \Delta\alpha] = [b_i + d_i(1-(\alpha+\Delta\alpha))]. \quad \square$$

Proposition 2. *Consider* $\theta = \lambda(x(\theta)) \in (0, 1)$ *and* $\alpha' = \theta + \Delta\theta$, *with* $\Delta\theta$ *as previously defined. Let* $\alpha'' = \lambda(x(\alpha'))$. *Then,* $\forall\alpha \in (\theta, \alpha'']$, $x(\alpha')$ *is an optimal solution of* $\mathrm{P}(\alpha)$.

Proof. It is clear that $x(\alpha')$ is an optimal solution of $\mathrm{P}(\alpha)$, $\forall\alpha \in [\alpha', \alpha'']$, by means of the above corollary. Now, we will show it $\forall\alpha \in (\theta, \alpha')$.

Let us assume $\beta \in (\theta, \alpha')$. Then let $x(\beta)$ be an optimal 0–1 solution of $\mathrm{P}(\beta)$. If $\beta' = \lambda(x(\beta))$, then $\beta' \geqslant \beta > \theta$. As $\beta' = \lambda(x(\beta))$, $\exists i \in M$ such that it follows that $a_i x(\beta) = b_i + d_i(1-\beta') < b_i + d_i(1-\theta)$, $a_i(\beta)$ being an integer number. Moreover, from the integer nature of the coefficients it follows that $[b_i + d_i(1-\beta')] = b_i + d_i(1-\beta')$.

Then two possibilities arise:

(a) $[b_i + d_i(1-\theta)] = b_i + d_i(1-\theta)$. But, due to the integer nature of the coefficients, $b_i + d_i(1-\beta') < b_i + d_i(1-\theta)$ and, moreover, $d_i \cdot \Delta\theta \leqslant 1$. Then

$$b_i + d_i(1-\beta') \leqslant b_i + d_i(1-\theta) - d_i \cdot \Delta\theta = b_i + d_i(1-(\theta+\Delta\theta)) = b_i + d_i(1-\alpha') \Rightarrow \beta' \geqslant \alpha'.$$

(b) $[b_i + d_i(1-\theta)] \neq b_i + d_i(1-\theta)$. But as $d_i \cdot \Delta\theta \leqslant \{b_i + d_i(1-\theta)\}$, we have

$$b_i + d_i(1-\beta') \leqslant [b_i + d_i(1-\theta)] + \{b_i + d_i(1-\theta)\} - d_i \cdot \Delta\theta = b_i + d_i(1-\alpha') \Rightarrow \beta' \geqslant \alpha'.$$

Since $\beta' \geqslant \alpha'$ and $\alpha' > \beta$, it follows that $\alpha' \in [\beta, \beta']$. Therefore, from the corollary, $x(\beta)$ is an optimal solution of $\mathrm{P}(\omega)$, $\forall\omega \in [\beta, \beta']$, and $x(\beta)$ is also an optimal solution of $\mathrm{P}(\alpha')$. Thus, $c \cdot x(\beta) = c \cdot x(\alpha')$ and $x(\alpha')$ is an optimal solution of $\mathrm{P}(\omega)$, $\forall\omega \in [\beta, \beta']$. Therefore, $x(\alpha')$ is an optimal solution of $\mathrm{P}(\beta)$, $\forall\beta \in (\theta, \alpha')$, and, finally, $x(\alpha')$ is an optimal solution of $\mathrm{P}(\alpha)$, $\forall\alpha \in (\theta, \alpha'']$. $\square$

These results are used to make an algorithm solving problem (12) using (13) as intermediate. The algorithm works as follows: First P(0), $\alpha_1 = 0$, is solved by using Glover's Enumeration Scheme [7]. Then the maximum value of α, say θ, for which the solution remains optimal in the interval $[\alpha_1, \theta]$ is determined. If $\theta \neq 1$, then consider $\alpha = \theta + \Delta\theta$, and solve $\mathrm{P}(\alpha)$ looking for a new interval of α. The process is repeated until $\theta = 1$ is reached.

The interval of α-values for which x is an optimal solution of $P(\alpha)$ will be denoted by $I_{x(\alpha)}$. Hence, it is clear that $I_{x(\alpha)} \subseteq [0, 1]$.

Thus, if an FBLP problem like (12) is assumed, the algorithm can be described as follows:

Step 0. Let $\alpha = \alpha_1 = 0$.
Step 1. Solve $\mathrm{P}(\alpha)$. Let $x(\alpha)$ be an optimal solution of $\mathrm{P}(\alpha)$.
Step 2. Let $\theta = \lambda(x(\alpha)) = \min\{\mu_i(x(\alpha)), i \in M\}$.
If $\alpha_1 = 0$, then $x(\alpha)$ is an optimal solution of $\mathrm{P}(\alpha)$, $\forall\alpha \in [\alpha_1, \theta]$;
else $x(\alpha)$ is an optimal solution of $\mathrm{P}(\alpha)$, $\forall\alpha \in (\alpha, \theta]$.

Step 3. If $\theta < 1$, then $\alpha_1 = \theta$, $\alpha = \theta + \Delta\theta$; go to Step 1.
Step 4. Stop.

3. Relating both solution methods

In the seminal paper by Bellman and Zadeh [1], one has an explicit fuzzy feasible set, $X \subset \mathbf{R}^n$, called a fuzzy constraint set, and an explicit fuzzy set of alternatives that attain a goal, $G \subset \mathbf{R}^n$, called a fuzzy goal.

The value of $\mu_G(x)$ indicates the degree to which $x \in X$ satisfies G. For example,

$$\mu_G(x) = \begin{cases} 1 & \text{for } f(x) \leqslant f^-, \\ f(x) & \text{for } f^- \leqslant f(x) \leqslant f^+, \\ 0 & \text{for } f(x) \geqslant f^+, \end{cases}$$

to be read as: We are fully satisfied ($\mu_G(x) = 1$) with the x's for which $f(x)$ attains a value lower than the aspiration level f^-; we are less satisfied (to degree $0 < f(x) < 1$) with the x's for which $f(x)$ is between f^+ and the lowest level f^-; finally, we are fully dissatisfied with the x's for which $f(x)$ is larger than f^-. And similarly for C.

This aspiration-level-based interpretation of G (and C) is not the only possible one, though it is certainly very convenient and intuitively appealing.

The problem is now stated as 'satisfy C and attain G', which by introducing a fuzzy set $D \subset \mathbf{R}^n$, the fuzzy set 'decision', can be written as

$$\mu_D(x) = \mu_C(x) \,\Delta\, \mu_G(x) = \min(\mu_C(x), \mu_G(x)) \quad \forall x \in X.$$

Though 'Δ' is the most commonly used operation, also here, other operations T, notably t-norms, may be used and then $\mu_D(x) = T(\mu_G(x), \mu_C(x))$. In [11] the use of the connective H_γ can be seen, which is an Archimedean t-norm, and corresponds to the intersection of fuzzy sets:

$$H_\gamma(\mu_A, \mu_B) = \frac{\mu_A \mu_B}{\gamma + (1-\gamma)(\mu_A + \mu_B - \mu_A \mu_B)}, \quad \gamma > 0, \tag{14}$$

where γ is an arbitrary parameter. It is evident that if $\gamma = 1$, then $H_\gamma(\mu_A, \mu_B) = \mu_A \cdot \mu_B$.

It should however be pointed out that problem (6) may become a nasty nonlinear programming problem for either nonlinear membership functions or other that the min-operator (see Zimmermann [13, pp. 100–108 and 254]).

The next problem is which $x \in X$ is to be chosen as a (nonfuzzy) solution to the problem. As commonly assumed, we seek an $x^* \in X$ such that

$$\mu_D(x^*) = \sup_{x \in X} \mu_D(x). \tag{15}$$

Proposition 3. *Let T be a* t-*norm, and μ_G and μ_C the respective membership functions of the fuzzy objective and the fuzzy constraint set. Then the following relation holds*:

$$\sup_\alpha T(\alpha, \max_{X(\alpha)} \mu_G(x)) = \max_{x \in X} T(\mu_G(x), \mu_C(x)). \tag{16}$$

Proof. Write

$$r = \max_{x \in X} T(\mu_G(x), \mu_C(x)) \quad \text{and} \quad r' = \sup_\alpha T(\alpha, \max_{X(\alpha)} \mu_G(x)).$$

Then it will be shown that $r = r'$.

Let $x' \in X$ be such that $T(\mu_G(x'), \mu_C(x')) = r$ and $\beta = \mu_C(x')$. It is clear that $x' \in X(\beta)$. Then

$$\sup_{\alpha} T(\alpha, \max_{X(\alpha)} \mu_G(x)) \geq T(\beta, \max_{X(\beta)} \mu_G(x)) \geq T(\mu_G(x'), \mu_C(x')) = r$$

and $r' \geq r$.

On the other hand, consider a fixed $\alpha \in [0, 1]$ and suppose the problem $\max_{X(\alpha)} \mu_G(x)$. Then, $\exists y \in X(\alpha)$ such that $\mu_G(y) = \max_{X(\alpha)} \mu_G(x)$ and therefore $T(\alpha, \max_{X(\alpha)} \mu_G(x) = T(\alpha, \mu_G(y))$. Thus, as $\mu_C(y) \geq \alpha$, $T(\mu_C(y), \mu_G(y)) \geq T(\alpha, \max_{X(\alpha)} \mu_G(x))$, and

$$\max_{X} T(\mu_G(x), \mu_C(x)) \geq \sup_{\alpha} T(\alpha, \max_{X(\alpha)} \mu_G(x)),$$

therefore $r \geq r'$, and consequently $r = r'$. □

According to Proposition 3, it is clear that $\mu_D(x^*) = \sup_\alpha T(\alpha, \max_{X(\alpha)} \mu_G(x)) = \alpha^*$ for x^* such that $\mu_G(x^*) = \sup_{X(\beta)} \mu_G(x)$ and $\beta = \mu_C(x^*)$.

It is evident, that if μ_G is an increasing (or decreasing) continuous function, then α^* always exists. Moreover, as the membership function of the fuzzy objective is linearly decreasing and the objective function is an increasing one, the following equivalence holds:

$$\max_{X(\alpha)} \mu_G(x) \Leftrightarrow \min_{X(\alpha)} cx.$$

Thus, if problem (4) has linear fuzzy constraints, then

$$X(\alpha) = \{x \in \mathbf{R}^n \mid x_j \in \{0, 1\}, \quad a_i x \leq b_i + d_i(1 - \alpha), \quad \alpha \in (0, 1], \quad i = 2, \ldots, m$$

and the alternative model to solve (4) is obtained as

$$\min \quad z = a_1 x \tag{17a}$$

$$\text{s.t.} \quad a_i x \leq b_i + d_i(1 - \alpha), \quad i = 2, \ldots, m \tag{17b}$$

$$x_j \in \{0, 1\}, \quad \alpha \in (0, 1]. \tag{17c}$$

If $x(\alpha)$ is the solution of (17), the following result shows that the maximizing decision provided by (14), or (5), can be obtained as a particular value of that parametric solution.

Proposition 4. *Let α^* be obtained from* (14), *or* (5), *and suppose that $x(\alpha)$ is the solution of* (17). *If $\{x_i(\alpha_i)\}$ denotes the set of points of the solution $x(\alpha)$, then*

$$\alpha^* = \max_{\{x_i(\alpha_i)\}} T(\mu_G(x_i(\alpha_i), \lambda(x_i(\alpha_i)). \tag{18}$$

Proof. Consider

$$\alpha^* = \max_{X} T(\mu_G(x), \mu_C(x)) = \sup_{\alpha} T(\alpha, \sup_{X(\alpha)} \mu_G(x)) \tag{19}$$

and x^* as the corresponding solution associated with problem (19). Writing $\beta = \mu_C(x^*)$, we have

$$\mu_G(x^*) = \sup_{X(\beta)} \mu_G(x) \geq \alpha^* \Rightarrow cx^* = \min_{X(\beta)} cx \Rightarrow cx^* = cx(\beta).$$

Thus, x^* is an optimal solution for $\mathrm{P}(\beta)$, and $\mu_G(x^*) = \mu_G(x(\beta))$. Therefore, as $\lambda(x(\beta)) = \mu_C(x(\beta))$, we obtain $\alpha^* = T(\mu_G(x(\beta)), \lambda(x(\beta)))$ and hence the optimal solution of (14) and (5) is obtained as

$$\alpha^* = \max_{\{x_i(\alpha_i)\}} T(\mu_G(x_i(\alpha_i), \lambda(x_i(\alpha_i)). \quad \square$$

The relation between the solutions of the above two approaches is illustrated by the following example.

4. Numerical example

Consider the same problem proposed in [12],

$$\begin{aligned} &\text{find } x_1, x_2, x_3 \in \{0, 1\} \\ &\text{s.t. } -10x_1 - 20x_2 - 20x_3 \lesssim -45, \\ &\quad x_1 + x_2 + x_3 \lesssim 2.5, \\ &\quad 2x_1 + x_2 + 3x_3 \lesssim 5, \\ &\quad 0.5x_1 + 3x_2 + x_3 \lesssim 3, \end{aligned}$$

in which the violations that the decision maker permits of the constraints are $d_1 = 10$, $d_2 = 0.75$, $d_3 = 1.5$ and $d_4 = 2$, respectively.

The solution provided to this problem in [12] is

$$\alpha^* = 0.5 \quad \text{and} \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 1.$$

On the other hand, (17) takes now the following form:

$$\begin{aligned} &\text{Min } z = -10x_1 - 20x_2 - 20x_3 \\ &\text{s.t. } x_1 + x_2 + x_3 \lesssim 2.5, \\ &\quad 2x_1 + x_2 + 3x_3 \lesssim 5, \\ &\quad 0.5x_1 + 3x_2 + x_3 \lesssim 3, \\ &\quad x_1, x_2, x_3 \in \{0, 1\}. \end{aligned}$$

Applying the above algorithm to this problem, the following solution is obtained:

$$\begin{aligned} &x(0) = (1, 1, 1), \quad I_{x(0)} = [0, 0.25], \\ &x(0.375) = (0, 1, 1), \quad I_{x(0.375)} = (0.25, 0.5], \\ &x(0.875) = (1, 0, 1), \quad I_{x(0.875)} = (0.5, 1], \end{aligned}$$

and therefore the corresponding fuzzy solution is $\underset{\sim}{S} = \{(1, 1, 1)/0.25, (0, 1, 1)/0.5, (1, 0, 1)1\}$. Hence, by applying Proposition 4, and using the minimum as t-norm, the above solution of [12] can be obtained:

$$\begin{aligned} &\mu_G(x(0.00)) = 1.0, \quad \lambda(x(0.00)) = 0.25, \\ &\mu_G(x(0.375)) = 0.5, \quad \lambda(x(0.375)) = 0.5, \\ &\mu_G(x(0.875)) = 0.0, \quad \lambda(x(0.875)) = 1. \end{aligned}$$

Thus, $\max_{\{x_i(\alpha_i)\}} (\lambda(x_i(\alpha_i)) \Delta \mu_G(x_i(\alpha_i)) = 0.5$ and therefore $\alpha^* = 0.5$ and $x^* = (0, 1, 1)$.

Final Remark

To face the solution of a Boolean programming problem with imprecisely stated constraints, that is an FBLP problem, a parametric Boolean programming problem has been proposed as an auxiliary model. Therefore the latter problem is a formal frame to find solutions to the former problem, and it has been shown that the solution of that parametric problem, which provides a fuzzy solution, involves as particular values the point solutions which could be obtained from other approaches.

References

[1] R.E. Bellman and L.A. Zadeh, Decision making in a fuzzy environment, *Management Science* **17** (1970) 141–164.

[2] M. Delgado, J.L. Verdegay and M.A. Vila, Optimization models in fuzzy-logic-based Decision Support Systems, in: R. Kulikowski and J. Rudnicki, Eds., *Proceedings of the Second Polish–Spanish Conference* (Omnitech Press, Warsaw, 1990) 35–46.

[3] F. Herrera and J.L. Verdegay, Approaching fuzzy integer linear programming problems, in: M. Fedrizzi, J. Kacprzyk and M. Roubens, Eds., *Interactive Fuzzy Optimization* (Springer–Verlag, Berlin, 1991) 78–91.

[4] R. Nauss and A.M. Geoffrion, Parametric and postoptimality analysis in integer linear programming, *Management Science* **23** (1977) 453–466.

[5] C.V. Negoita and D.A. Ralescu, *Applications of Fuzzy Sets to Systems Analysis* (Birkhauser, Stuttgart, 1975).

[6] S.A. Orlovski, On programming with fuzzy constraint sets, *Kybernetes* **6** (1977) 197–201.

[7] H.A. Taha, *Integer Programming. Theory, Applications and Computations* (Academic Press, New York, 1975).

[8] J.L. Verdegay, Fuzzy mathematical programming, in: M.M. Gupta and E. Sanchez, Eds., *Fuzzy Information and Decision Processes* (North-Holland, New York, 1982) 231–237.

[9] R.R. Yager, A Mathematical programming approach to inference with the capability of implementing default rules, *Int. J. of Man–Machine Studies* **29** (1988) 685–714.

[10] H.-J. Zimmermann, Description and optimization fuzzy systems, *International Journal of General Systems* **2** (1976) 209–215.

[11] H.-J. Zimmermann, Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems* **1** (1978) 45–55.

[12] H.-J. Zimmermann and M.A. Pollatschek, Fuzzy 0–1 linear programs, in: H.-J. Zimmermann, L.A. Zadeh and B.R. Gaines, Eds., *Fuzzy Sets and Decision Analysis* (North-Holland, New York, 1984) 133–145.

[13] H.-J. Zimmermann, *Fuzzy Sets, Decision Making, and Expert Systems* (Kluwer Academic Publishers, Boston, 1987).

[14] H.-J. Zimmermann, *Fuzzy Set Theory and its Applications*, 2nd edn. (Kluwer Academic Publishers, Boston, 1991).