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Post-optimality analysis on the membership functions of a fuzzy linear programming problem*

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Abstract: Models of linear programming problems with fuzzy constraints are very well known in the current literature. In almost all cases, to solve these problems, linear membership functions are used because they have very good properties and are very easy to manipulate. In some cases, however, because of the knowledge that the decision maker has, such membership functions could be modeled as nonlinear, although the complexity of the problem could increase. This paper considers the use of nonlinear membership functions in fuzzy linear programming problems to show that the corresponding solution to be obtained can be derived from a parallel linear model. Moreover, it is easier to solve than the nonlinear model, making use of a similar procedure to that of post-optimal analysis in classical linear programming. The case in which these membership functions are defined by means of piecewise linear approximations is also considered and analyzed.

Keywords: Mathematical programming; membership functions; fuzzy constraints.

1. Introduction

In general a linear programming (LP) problem is described by

$$\mathrm{Max}\{cx \mid Ax \leqslant b,\, x \geqslant 0\}$$

where A is a $m \times n$ matrix of real numbers, $b \in \mathbf{R}^m$ and $c \in \mathbf{R}^n$.

It is clear in this case the decision maker is assumed to have complete and precise information about all the elements taking part into the problem. Frequently, however, the decision maker prefers to express the parameters defining the problem in a linguistic way rather than in a numerical and exact one.

From this fact, and based upon the seminal paper by Bellman and Zadeh [1], the first approaches to fuzzy linear programming (FLP) were made [12, 15]. The starting model, parallel to the above one, is an optimization problem in which the constraints are defined by respective fuzzy sets, which can be represented by

$$\mathrm{Max}\{cx \mid Ax \lesssim b,\, x \geqslant 0\}.$$

Here, one permits the decision maker to accept moderate violations in the satisfying of the constraints. Such violations are evaluated by means of the corresponding membership function associated to each constraint,

$$\mu_i : \mathbf{R} \to [0, 1], \quad \forall i \in M,\, M = \{1, 2, \ldots, m\}.$$

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It is very important to note that one assumes that the decision maker is able to define these membership functions with exactness, which is similar to the previous assumption about the precise knowledge that the decision maker has regarding the precise specification of the values of the parameters.

From this initial, soft approach, contributions to the topic of FLP have been addressed mainly from four different, but related view points:

(a) Models incorporating concepts and results from fuzzy arithmetic. For example, we can assume that the parameters either in the technological matrix, the costs or the right hand side are fuzzy numbers.

(b) Methods providing new ways of solving the different models. This is the case, for instance, when we apply classical parametric techniques based on the representation theorem for fuzzy sets.

(c) Extensions of the above general model to more complex problems. The main focus has been on multiobjective problems, but transportation problems and games have been topics also widely studied.

(d) Applications to concrete practical problems.

There is, as said before, in all these contributions a common hypothesis: the fuzzy sets taking part into the model have membership functions very precisely defined by the decision maker. This means that is impossible to introduce any changes, e.g., on their shapes or margins, without changing completely the formulation of the problem. However, to study in what ways solutions can change according to changes in the membership functions should be a very interesting problem. More concretely, the problem to be treated is that of the sensitivity of the membership functions. The references on this problem in the literature are [3] and [6]. In [6], like in the classical case, the main focus is on the sensitivity of the right hand side, and in [3] a different conception about violation margins and the membership functions representing the fuzzy constraints have been considered.

This paper develops a solution approach to FLP problems with nonlinear membership functions. Suppose an FLP problem with fuzzy constraints, that is, with a membership function for each constraint. Let x^* be the optimal fuzzy solution. It is clear if these membership functions are changed, then x^* will change too. But can the new optimal fuzzy solution be obtained from the former one?

With regard to this problem, the next section will introduce briefly the FLP problem with linear membership functions. Section 3 is devoted to the use of nonlinear membership functions to model fuzzy constraints. Section 4 shows this approach with piecewise approximation of the nonlinear membership functions. Finally, to show the effectiveness of the proposed approach, a numerical example is analyzed and some conclusions are pointed out.

2. FLP problems with linear membership functions

Consider a decision maker faced with an LP problem in which he can tolerate violations in the accomplishment of the constraints, that is, he permits the constraints to be satisfied as well as possible. For each constraint in the constraint set this assumption can be represented by

$$a_i x \lesssim b_i, \quad \forall i \in M \tag{1}$$

and, for every i, modeled by means of a membership function

$$\mu_i(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b_i, \\ f_i(a_i x) & \text{if } b_i \leqslant a_i \leqslant b_i + d_i, \\ 0 & \text{if } a_i x \geqslant b_i + d_i, \end{cases} \tag{2}$$

where $f_i(\cdot)$ is strictly decreasing and continuous for $a_i x$, $f_i(b_i) = 1$ and $f_i(b_i + d_i) = 0$.

This membership function expresses that the decision maker tolerates violations in the accomplishment of the constraint i up the value $b_i + d_i$. The function $\mu_i(\cdot)$ gives the degree of satisfaction of the i-th constraint for $x \in \mathbf{R}^n$, but this value is obtained by means of the function f_i which are defined over $\mathbf{R}$.

With these assumptions in mind, the associated FLP problem can be presented as

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x \geqslant 0, \end{aligned} \tag{3}$$

where $c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$ and A is an $m \times n$ matrix of real numbers.

Problem (3) has been described in [12] and [15], where some additional hypotheses about the fuzzy nature of the objective function, that are not relevant here, are also considered.

With regard to solving (3), three different approaches can be considered [12, 13, 15]. In particular, making use of the representation theorem for fuzzy sets, [13] shows how to find a fuzzy solution to (3) by means of the auxiliary parametric LP problem

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & Ax \leqslant g(\alpha), \\ & x \geqslant 0, \quad \alpha \in (0, 1], \end{aligned} \tag{4}$$

where $g(\alpha) \in \mathbf{R}^m$ is a column vector defined by the inverse functions of the f_i. The fuzzy solution involves as particular values those solutions proposed in [12] and [15]. In particular, if f_i are linear functions, then the membership functions are

$$\mu_i(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b_i, \\ \dfrac{b_i + d_i - a_i x}{d_i} & \text{if } b_i \leqslant a_i x \leqslant b_i + d_i, \\ 0 & \text{if } a_i x \geqslant b_i + d_i, \end{cases} \tag{5}$$

and (3) becomes

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & a_i x \leqslant b_i + d_i(1 - \alpha), \quad \forall i \in M, \\ & x \geqslant 0, \quad \alpha \in (0, 1]. \end{aligned} \tag{6}$$

Hence, if $x(\alpha)$ is the optimal parametric solution of (6), the fuzzy solution for (3) will be the fuzzy set $\{x(\alpha)/\alpha\}$, which will be denoted by $x(\alpha)$.

In some cases however the decision maker can prefer to express his satisfaction degrees on the constraints by means of nonlinear membership functions, perhaps more complex than the above linear ones. In this case, a method that employs conventional parametric techniques based on the representation theorem, is also able of solving the FLP problem.

The next section focuses on this nonlinear case, generating a method to relate its corresponding optimal fuzzy solution with that one obtained when the problem is solved by using linear membership functions.

3. FLP problems with nonlinear membership functions

Consider (3), with the fuzziness of the constraints now represented by nonlinear membership functions

$$\mu_i'(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b_i, \\ f_i'(a_i x) & \text{if } b_i \leqslant a_i x \leqslant b_i + d_i, \quad i \in M, \\ 0 & \text{if } a_i x \geqslant b_i + d_i, \end{cases} \tag{7}$$

where $f_i'(\cdot)$ is strictly decreasing and continuous for $a_i x$, $f_i'(b_i) = 1$ and $f_i'(b_i + d_i) = 0$. These functions can be graphically represented as in Figure 1.

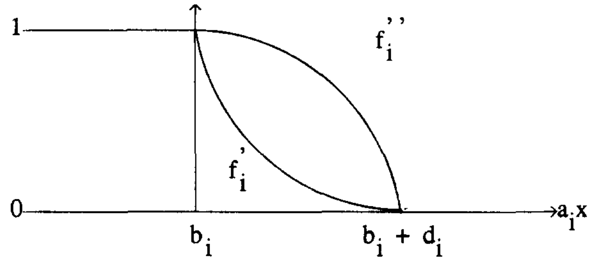

Fig. 1.

Then, using the approach proposed in [13], and in a parallel way to the discussion above for the linear case, the fuzzy optimal solution for (3) can be obtained from the optimal parametric solution of the problem

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & Ax \leqslant g'(\alpha), \\ & x \geqslant 0, \quad \alpha \in (0, 1], \end{aligned} \tag{8}$$

where $g'(\alpha) = f'^{-1}(\alpha)$, $\forall \alpha \in (0, 1]$.

We now try to find some link relating the two solutions obtained from the linear case and from the nonlinear one. To solve it, the following results will be needed.

Proposition 1. *Let $[a, b]$ be a real interval and $f:[a, b] \to [0, 1]$ be any continuous, strictly decreasing linear function, such that $f(a) = 1$ and $f(b) = 0$. For every other continuous, strictly decreasing function $f':[a, b] \to [0, 1]$, such that $f'(a) = 1$ and $f'(b) = 0$, there exists a function $r:[0, 1] \to [0, 1]$ such that $r(\cdot) \circ f(\cdot) = f'(\cdot)$.*

Proof. As f is a continuous strictly decreasing linear function in $[a, b]$, it is possible to define its inverse, $f^{-1}(\cdot)$, in $[0, 1]$. Then, the function r exists and is defined as $r(\cdot) = f'(\cdot) \circ f^{-1}(\cdot)$. □

Applying this proposition to the nonlinear function $f_i'(\cdot)$ and the linear function $f_i(\cdot)$, associated to the membership functions $\mu_i'(\cdot)$ and $\mu_i(\cdot)$ respectively, it is clear that there exists a function $r:[0, 1] \to [0, 1]$ such that $r(\cdot) = f_i'(\cdot) \circ f_i^{-1}(\cdot)$ and $\mu_i'(\cdot) = r(\cdot) \circ \mu_i(\cdot)$. If $\mu_i'(\cdot) = r(\cdot) \circ \mu_i(\cdot)$ $\forall i \in M$, which is not restrictive because it may be reasonable for the decision maker to propose the same type of membership function for every constraint, then we obtain the following result.

Proposition 2. *Consider the* FLP *problem* (3). *Denote by $x(\cdot)$ and $x'(\cdot)$ the optimal fuzzy solution for this problem, using linear and nonlinear membership functions repsectively to model the fuzziness of the constraints. Then, $x'(\alpha) = x(r^{-1}(\alpha))$, where $r(\cdot)$ is obtained from Proposition* 1.

Proof. Let $x'(\alpha)$ be the solution of the FLP problem with nonlinear membership functions and the auxiliary parametric LP problem

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & a_i x \leqslant f_i'^{-1}(\alpha), \\ & x \geqslant 0, \quad \alpha \in (0, 1], \end{aligned}$$

where $f_i'(\cdot)$ is the function associated to the membership function $\mu_i'(\cdot)$, $i \in M$.

Let now $f_i(\cdot)$ be the linear function associated to the membership function $\mu_i(\cdot)$. Then, from Proposition 1, there exists another function $r(\cdot)$ such that $r \circ f_i'$. Thus,

$$f_i'^{-1}(\alpha) = (r \circ f_i)^{-1}(\alpha) = f_i^{-1}(r^{-1}(\alpha)).$$

Denoting $\beta = r^{-1}(\alpha)$, this expression can be rewritten as $f_i'^{-1}(\alpha) = f_i^{-1}(\beta)$. Hence, the FLP with linear membership functions may be rewritten as

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & a_i x \leqslant f_i^{-1}(\beta), \\ & x \geqslant 0, \quad \beta \in (0, 1], \end{aligned}$$

from which an optimal solution $x(\beta)$ can be obtained.

These two ('linear' and 'nonlinear') problems are formally identical. Moreover, one has

$$x'(\alpha) = x(\beta) = x(r^{-1}(\alpha)). \qquad \square$$

Therefore, the corresponding value of the objective function will be derived according to

$$z'(\alpha) = c \cdot x'(\alpha) = c \cdot x(r^{-1}(\alpha)).$$

This shows that solving an FLP problem with constraints modeled by linear membership functions we can obtain the optimal fuzzy solution for the same problem modeled by nonlinear membership functions. Then it verifies the conditions of the above proposition: It suffices to use the above function $r(\cdot)$, which can be easily obtained.

4. Piecewise linear membership functions

Different paper [7, 8, 14] have formulated FLP with all the membership functions modeling the fuzzy constraints given in piecewise linear forms.

Suppose μ_i, $\forall i \in M$, is a continuous piecewise linear membership function. A function with this characteristic can be expressed as

$$\mu_i(x) = \lim_{j=1,\dots,N_i+1} (t_{ij} \cdot a_j \cdot x + s_{ij}) \tag{9}$$

in the range $(0, 1)$. It is assumed that $\mu_i(x) = t_{ij} \cdot a_i \cdot x + s_{ij}$ for each segment $g_{ij-1} \leqslant a_i x \leqslant g_{ij}$ $j = 1, \dots, N_i + 1$. That is, t_{ij} is the slope and s_{ij} is the y-intercept for the section of the curve starting at g_{ij-1} and ending at g_{ij}.

Graphically, a membership function shaped as in Figure 2 is being considered.

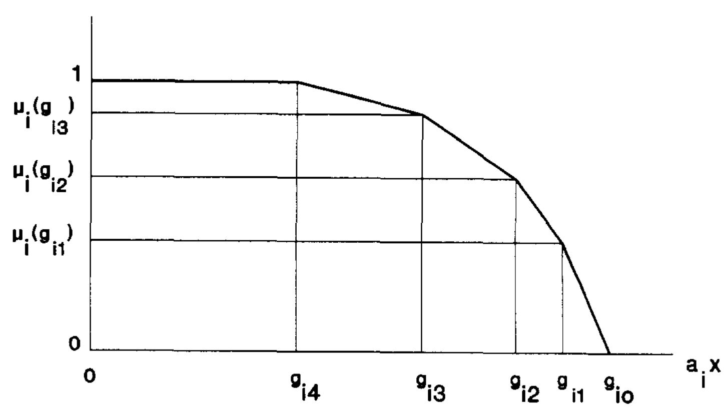

Fig. 2.

where $g_{i0} = b_i + d_i$ and $g_{i4} = b_i$. This kind of membership functions may be defined by

$$\mu_i''(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b_i, \\ f_i''(a_i x) & \text{if } b_i \leqslant a_i x \leqslant b_i + d_i, \\ 0 & \text{if } a_i x \geqslant b_i + d_i. \end{cases} \tag{10}$$

Each segment is defined as $g_{ij-1} \leqslant a_i x \leqslant g_{ij}$ such that $g_{i0} = b_i + d_i$ and $g_{iN_i+1} = b_i \; \forall i \in M$, $j = 1, 2, \ldots, N_i + 1$, and $f_i''(\cdot)$ is a continuous strictly decreasing function given by

$$f_i''(a_i x) = f_{ij}''(a_i x) = t_{ij} a_i x + s_{ij}. \tag{11}$$

As this function satisfies the conditions for Proposition 1, we know that there exists a strictly increasing function $h : [0, 1] \to [0, 1]$ such that $f_i''(\cdot) = h(\cdot) \circ f_i(\cdot)$, and it is defined by

$$h(\cdot) = f_{ij}'' \circ f_i^{-1}(\cdot) \quad \text{if } g_{ij-1} \leqslant f_i^{-1}(\cdot) \leqslant g_{ij}, \; j = 1, \ldots, N_i + 1.$$

Therefore, if $\mu_i''(\cdot) = h(\cdot) \circ \mu_i(\cdot) \; \forall i \in M$, then, by the Proposition 2, the fuzzy solution for (3), denoted by $x''(\alpha)$, when piecewise linear membership functions are used to model the fuzzy constraints, can be obtained as

$$x''(\alpha) = x(h^{-1}(\alpha)).$$

Otherwise, the fuzzy solution will be obtained solving the following parametric LP problem:

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & a_i x \leqslant f_i''^{-1}(\alpha), \quad i \in M, \\ & x \geqslant 0, \quad \alpha \in (0, 1], \end{aligned} \tag{12}$$

where $f_i''^{-1}(\cdot)$ are piecewise linear functions.

5. Numerical example

Consider the following FLP problem:

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leqslant 3, \\ & x_1 + x_2 \leqslant 5, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \end{aligned} \tag{13}$$

for which the decision maker permits violations $d_1 = 3.5$ and $d_2 = 6.5$ of the constraints respectively.

In fact, if the decision maker defines linear membership functions, the auxiliary model to be solved is the following parametric LP problem:

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leqslant 3 + 3.5(1 - \alpha), \\ & x_1 + x_2 \leqslant 5 + 6.5(1 - \alpha), \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad \alpha \in (0, 1], \end{aligned} \tag{14}$$

whose optimal solution is obtained as

$$x_1(\alpha) = 1.6666 - \alpha, \qquad x_2(\alpha) = 7.6666 - 3.3333\alpha,$$

and the value of the objective function

$$z(\alpha) = 2x_1(\alpha) + 3x_2(\alpha) = 26.3333 - 12\alpha.$$

Suppose that in order to solve this problem, the decision maker proposes to model the fuzzy constraints by means of nonlinear membership functions.

Assume first, parabolic concave membership functions defined by

$$\mu_i'(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b, \\ -\dfrac{1}{d_i^2}(a_i x)^2 + \dfrac{2b_i}{d_i^2}(a_i x) + 1 - \dfrac{b_i^2}{d_i^2} & \text{if } b_i \leqslant a_i x \leqslant b_i + d_i, \\ 0 & \text{if } a_i x \geqslant b_i + d_i. \end{cases}$$

Then, the fuzzy solution is obtained solving the following parametric LP problem:

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leqslant 3 + 3.5\sqrt{(1-\alpha)}, \\ & x_1 + x_2 \leqslant 5 + 6.5\sqrt{(1-\alpha)}, \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad \alpha \in (0, 1]. \end{aligned} \tag{15}$$

By Proposition 1, $r(\beta) = 1 - (1-\beta)^2$ and $r^{-1}(\alpha) = 1 - \sqrt{1-\alpha}$, and the optimal solution by Proposition 2 is

$$x_1'(\alpha) = x_1(r^{-1}(\alpha)) = 1.6666 - (1 - \sqrt{1-\alpha}) = 0.6666 + \sqrt{1-\alpha},$$
$$x_2'(\alpha) = x_2(r^{-1}(\alpha)) = 7.6666 - 3.3333(1 - \sqrt{1-\alpha}) = 4.3333 + 3.3333\sqrt{1-\alpha},$$

and the corresponding optimal value of the objective is

$$z'(\alpha) = 14.3333 + 12\sqrt{1-\alpha}.$$

Now, suppose the decision maker uses exponential convex membership functions defined by

$$\mu_i'(x) = \begin{cases} 1 & \text{if } a_i x \leqslant b_i \\ \dfrac{q^{t\cdot(a_i x - b_i)/d_i} - q^t}{1 - q^t} & \text{if } b_i \leqslant a_i x \leqslant b_i + d_i, \ 0 < q < 1, \ t > 0, \\ 0 & \text{if } a_i x \geqslant b_i + d_i \end{cases}$$

If q and t are fixed, for example $q = 0.5$ and $t = 1$, we can obtain the fuzzy solution solving the following auxiliary problem:

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leqslant 3 + 3.5\log_{0.5}(0.5 + 0.5\alpha), \\ & x_1 + x_2 \leqslant 5 + 6.5\log_{0.5}(0.5 + 0.5\alpha), \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad \alpha \in (0, 1], \end{aligned} \tag{16}$$

and by Proposition 1,

$$0 < q < 1, \quad t > 0, \quad r^q(\beta) = \frac{q^{t(1-\beta)} - q^t}{1 - q^t} \text{ and } r_q^{-1}(\alpha) = \frac{t - \log_q(q^t + \alpha(1 - q^t))}{t}$$

Its optimal solution is

$$x_1'(\alpha) = 0.6666 + \log_{0.5}(0.5 + 0.5\alpha), \qquad x_2'(\alpha) = 4.3333 + 3.3333\log_{0.5}(0.5 + 0.5\alpha)$$

and the value of the objective function is

$$z'(\alpha) = 14.3333 + 12\log_{0.5}(0.5 + 0.5\alpha).$$

Consider, finally, the decision maker uses the following piecewise membership functions:

$$\mu_1''(x)=\begin{cases}1 & \text{if } a_i x 3,\\ (10-(-2x_1+x_2))/7 & \text{if } 3\leqslant a_i x\leqslant 5.1,\\ (6.5-(-2x_1+x_2))/2 & \text{if } 5.1\leqslant a_i x\leqslant 6.5,\\ 0 & \text{if } a_i x\geqslant 6.5,\end{cases}$$

$$\mu_2''(x)=\begin{cases}1 & \text{if } a_i x\leqslant 5,\\ (18-(-2x_1+x_2))/13 & \text{if } 5\leqslant a_i x\leqslant 8.9,\\ (11.5-(-2x_1x_2))/3.714 & \text{if } 8.9\leqslant a_i x\leqslant 11.5,\\ 0 & \text{if } a_i x\geqslant 11.5.\end{cases}$$

Then, the corresponding fuzzy solution for the former problem will be obtained solving the two following parametric LP problems:

$$\begin{array}{llll} \text{Max} & z=2x_1+3x_2 & \text{Max} & z=2x_1+3x_2\\ \text{s.t.} & -2x_1+x_2\leqslant 6.5-2\alpha, & \text{s.t.} & -2x_1+x_2\leqslant 10-7\alpha,\\ & x_1+x_2\leqslant 11.5-13\alpha/3.5, & & x_1+x_2\leqslant 18-13\alpha,\\ & x_1\geqslant 0,\quad x_2\geqslant 0,\quad \alpha\in(0,0.7], & & x_1\geqslant 0,\quad x_2\geqslant 0,\quad \alpha\in(0.7,1], \end{array}\tag{17}$$

with

$$h(\beta)=\begin{cases}(3.5+3.5\beta)/7 & \text{if } \beta\geqslant 0.4,\\ 3.5\beta/2 & \text{if } \beta\leqslant 0.4,\end{cases}\quad \text{and}\quad h^{-1}(\alpha)=\begin{cases}2\alpha-1 & \text{if } \alpha\geqslant 0.7,\\ 2\alpha/3.5 & \text{if } \alpha\leqslant 0.7,\end{cases}$$

and whose optimal solutions are respectively

$$x_1''(\alpha)=1.6666-2\alpha/3.5 \quad\text{and}\quad x_2(\alpha)=7.6666-6.6666\alpha/3.5 \quad \text{if } \alpha\leqslant 0.7,$$
$$x_1''(\alpha)=2.6666-2\alpha \quad\text{and}\quad x_2(\alpha)=11-6.6666\alpha \quad \text{if } \alpha\geqslant 0.7.$$

In this case, the object function takes the value

$$z''(\alpha)=\begin{cases}26.3333-24\alpha/3.5 & \text{if } \alpha\leqslant 0.7,\\ 38.3333-24\alpha & \text{if } \alpha\geqslant 0.7.\end{cases}$$

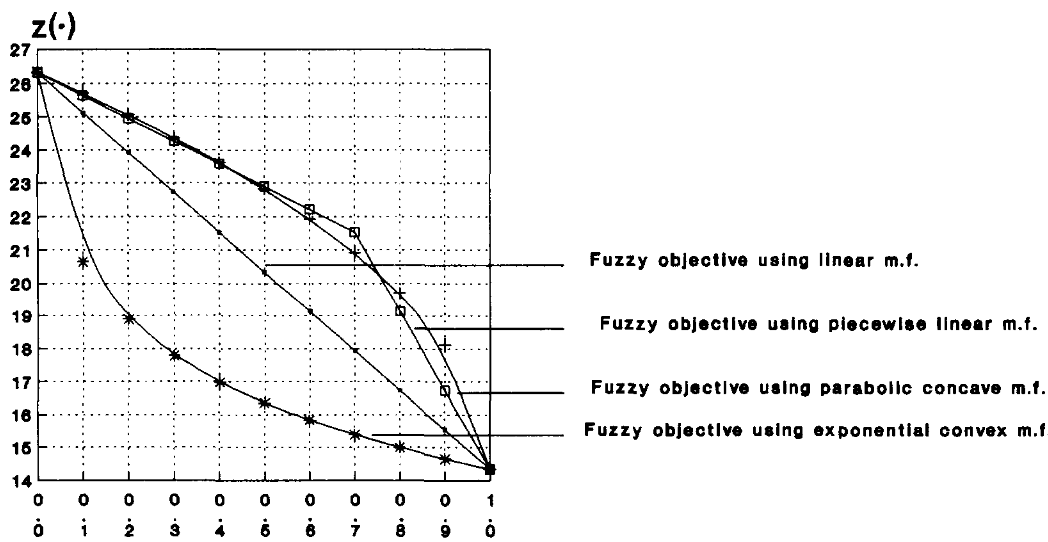

Fig. 3.

Remark. Notice that the functions obtained by means of the Proposition 1 and called $r(\cdot)$ or $h(\cdot)$ allow us to obtain the solution of the problems (15), (16) and (17) by applying Proposition 2.

With respect to the objective functions, each of them can be easily obtained and represented as in Figure 3.

Conclusions

Linear programming (LP) problems with fuzzy constraints have been considered, and the case of representing these restrictions by nonlinear membership functions analyzed. It has been shown that we can obtain the fuzzy optimal solution from the solution of a parallel LP problem modeled by linear membership functions when the nonlinear membership functions verify the conditions of the above propositions. This approach includes the case of piecewise linear functions.

The way of obtaining the solution can be seen as analogous to a post-optimal analysis on the membership functions, which shows that only linear membership functions are needed to determine the solution for nonlinear membership functions.

The results obtained here also allow us to solve parametric LP problems where the right hand side is formulated by means of nonlinear parametric functions and these verify the conditions of Proposition 1.

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