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AN OPTIMUM CONCEPT FOR FUZZIFIED LINEAR PROGRAMMING PROBLEMS: A PARAMETRIC APPROACH

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ABSTRACT. In this paper the optimality concept for (g,p)-fuzzified linear programming problems is studied. It is shown that this model can be solved by means of parametric linear programming problems. Moreover, some results about the (g,p)-fuzzified linear programming problem are obtained using the parametric linear programming problem.

1 Introduction

Let us consider the classical linear programming problem

$$\sum_{j=1}^{n} \gamma_j x_j \longrightarrow \min \tag{1}$$

subject to

$$\sum_{j=1}^{n} \alpha_{ij} x_j \le \alpha_{i0}, \quad i = 1, \dots, s, \tag{2}$$

$$x_j \ge 0, \quad j = 1, \dots, n, \tag{3}$$

where $\alpha_{ij}, \gamma_j \in \mathbb{R}$, $i = 1, \dots, s$, $j = 0, 1, \dots, n$.

In the problem (1)–(3) the coefficients α_{ij} and γ_j are supposed to be well defined. However, these coefficients are generally known only approximately.

In this paper instead of (1)–(3) we will examine the fuzzified version of this problem assuming that the coefficients in the problem formulation are given by

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fuzzy numbers and the relations in the definition of the feasible set are also fuzzy relations. Specially, we will examine the so called (g,p)-fuzzified linear programming problems [3,6]. In this approach the side function g of the fuzzifying parameters and the generator function g^p of the Archimedean t-norm used in the extension principle and in the intersection and Cartesian product of fuzzy sets are defined by the same function g. We present an alternative formulation of the optimality concept presented by Kovács [4,5]. This formulation is based on the parametric programming. The parametric approach has been studied in [1,2,7].

2 Preliminaries

Let X be a universe and I be the unit interval on the real line $\mathbb{R}$. Let $\mathcal{F}(X) = \{\mu \mid \mu : X \to I\}$ denote the set of all fuzzy subsets on X. The characteristic function of $A \subset X$ is a special fuzzy subset, which will be denoted by χ_A.

A binary operation $T : I \times I \to I$ is said to be a t-norm iff it is commutative, associative, non-decreasing and $T(a,1) = a$ for all $a \in I$. A t-norm is Archimedean if it is continuous and $T(a,a) < a$ for all $0 < a < 1$. An Archimedean t-norm admits the representation $T(a,b) = g^{(-1)}\big(g(a) + g(b)\big)$, where the generator function $g : I \to \mathbb{R}_+$ is continuous, strictly decreasing, $g(1) = 0$ and $g^{(-1)}(x) = g^{-1}(x)$ if $x \in [0, g(0)]$ and $g^{(-1)}(x) = 0$ if $x > g(0) = g_0$ ($g_0 = \infty$ is also allowed). Every t-norm induces an n-ary operation $T_{n-1} : I^n \to I$ with the following rule $T^{n-1}(a_1, \ldots, a_n) = T\big(T^{n-2}(a_1, \ldots, a_{n-1}), a_n\big)$. If T is Archimedean, then $T^{n-1}(a_1, \ldots, a_n) = g^{(-1)}\big(\sum_{j=1}^{n} g(a_j)\big)$.

Let $g : I \to \mathbb{R}_+$ be a fixed function with the properties of a generator function and let $\mathcal{F}_g$ denote the subset of $\mathcal{F}(\mathbb{R})$ (set of fuzzy numbers over $\mathbb{R}$) containing the fuzzy numbers with the membership function

$$\mu(a) = \begin{cases} g^{(-1)}(|a - \alpha|/d) & \text{if } d > 0, \\ \chi_{\{\alpha\}}(a) & \text{otherwise,} \end{cases} \tag{4}$$

for all $\alpha \in \mathbb{R}$, $d \in \mathbb{R}_+ \cup \{0\}$. The elements of $\mathcal{F}_g$ will be called quasitriangular fuzzy numbers generated by g with the center α and spread d, and we will refer to them by the pair (α, d). Let d be a fixed value and let $\mathcal{F}_{gd}$ denote the subset of $\mathcal{F}_g$ containing the quasitriangular fuzzy numbers with the same spread d.

Let T_{gp} be an Archimedean t-norm given by the generator function g^p, $1 \leq p \leq \infty$. It is easy to see that $\lim_{p\to\infty} T_{gp}(a,b) = \min(a,b)$, therefore we will also use the notation T_{gp} in the case $p=\infty$ meaning the min-norm for $T_{g\infty}$.

Let $\mu_j = (\alpha_j, d_j) \in \mathcal{F}_g$, $j = 1,\dots,n$. The T_{gp}-Cartesian product $\mu_a = \mu_1 \times \dots \times \mu_n$ of these n quasitriangular fuzzy numbers generated by the same g, will be called (g,p)-fuzzy vector on $\mathcal{F}_g^n$. Shortly we will use the notation $\mu_a = (\boldsymbol{\alpha}, \boldsymbol{d}) \in \mathcal{F}_g^n$. If $d_1 = \dots = d_n = d$, then $\mu_a = (\alpha, d) \in \mathcal{F}_{gd}^n$. In this case it is easy to show that

$$\mu_a(\boldsymbol{a}) = \mu_a(a_1,\dots,a_n) = \begin{cases} g^{(-1)}(\|\boldsymbol{a}-\boldsymbol{\alpha}\|_p/d) & \text{if } d \neq 0, \\ \chi_{\{\alpha_1,\dots,\alpha_n\}}(a_1,\dots,a_n) & \text{if } d = 0, \end{cases}$$

where

$$\|\boldsymbol{a}\|_p = \begin{cases} \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{j=1,\dots,n} |a_j| & \text{if } p = \infty. \end{cases} \tag{5}$$

3 Previous results

In this section we will study the (g,p)-fuzzified linear programming problem and a possible optimum concept for this fuzzified problem. Our discussion will be based on the results of papers [3,4,5,6].

At every fixed $\boldsymbol{x} \in \mathbb{R}^n$ the (g,p)-fuzzified function value of the parametrical function $f(\boldsymbol{a},\boldsymbol{x}) = f(a_1,\dots,a_n,\boldsymbol{x})$ is a fuzzy set on $\mathbb{R}$ of which the membership function is obtained by the extension principle changing the parameter vector $\boldsymbol{a}$ in the function $f(\boldsymbol{a},\boldsymbol{x})$ with the (g,p)-fuzzy parameter vector $\mu_a = (\boldsymbol{\alpha}, \boldsymbol{d}) \in \mathcal{F}_g^n$. Namely, the membership function of the (g,p)-fuzzified function value is defined by

$$\tilde{f}(\mu_a, \boldsymbol{x})(y) = \begin{cases} \sup_{\boldsymbol{a}\in A(\boldsymbol{x},y)} \mu_a(\boldsymbol{a}) & \text{if } A(\boldsymbol{x},y) \neq \emptyset, \\ 0 & \text{if } A(\boldsymbol{x},y) = \emptyset, \end{cases}$$

where $A(\boldsymbol{x},y) = \{\boldsymbol{a} = (a_1,\dots,a_n) \mid a_i \in \mathbb{R},\ y = f(\boldsymbol{a},\boldsymbol{x})\}$.

Thus, let the linear function $\ell(\boldsymbol{\alpha},\boldsymbol{x}) = \sum_{j=1}^{n} \alpha_j x_j$ be fuzzified by the (g,p)-fuzzy parameter vector $\mu_a = (\boldsymbol{\alpha}, d) = \mu_1 \times \dots \times \mu_n \in \mathcal{F}_{gd}^n$, $\mu_j = (\alpha_j, d) \in \mathcal{F}_{gd}$, $j = 1,\dots,n$. If $d > 0$, then the (g,p)-fuzzified linear function value is the fuzzy number $\tilde{\ell}_g(\boldsymbol{\mu}_a, \mathbf{x}) = (\sum_{j=1}^{n} \alpha_j x_j, D(\mathbf{x})) \in \mathcal{F}_g$, i.e.

$$\tilde{\ell}_g(\boldsymbol{\mu}_a, \mathbf{x})(y) = \begin{cases} g^{(-1)}(|y - \sum_{j=1}^{n} \alpha_j x_j|/D(\mathbf{x})) & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \chi_{\{0\}}(y) & \text{if } \mathbf{x} = \mathbf{0}, \end{cases}$$

where $D(\mathbf{x}) = d\|\mathbf{x}\|_q$ and $\|\mathbf{x}\|_q$ is computed according to (5) with

$$q = \begin{cases} 1 & \text{if } p = \infty, \\ \infty & \text{if } p = 1, \\ p/(p-1) & \text{otherwise.} \end{cases}$$

Let R be an unfuzzy relation on the real line. For every $\mu, \nu \in \mathcal{F}_g$ the T_{gp}-fuzzification $\tilde{R}$ of R on $\mathcal{F}_g$ is a fuzzy set on $\mathcal{F}_g \times \mathcal{F}_g$ defined by

$$\tilde{R}(\mu, \nu) = \sup_{xRy} T_{gp}(\mu(x), \nu(y)).$$

Let us use this definition for the fuzzified function values and the right hand sides of the constraint inequalities.

Let the i-th relation in (2) fuzzified by the (g,p)-fuzzy parameter vector $\boldsymbol{\mu}_i = (\boldsymbol{\alpha}_i, d) = \mu_{i0} \times \dots \mu_{in} \in \mathcal{F}_{gd}^{n+1}$, $\mu_j = (\alpha_j, d) \in \mathcal{F}_{gd}$, $j = 0, 1, \dots, n$. If $d > 0$, then the (g,p)-fuzzified i-th constraint defines the fuzzy constraint with the membership function

$$\sigma_i(\mathbf{x}) = g^{(-1)}(\max(0, \sum_{j=1}^{n} \alpha_{ij} x_j - \alpha_0)/D_0(\mathbf{x})),$$

where $D_0(\mathbf{x}) = d\|(\mathbf{x}, 1)\|_q$ and $\|\mathbf{x}\|_q$ and q are defined above.

The fuzzy feasible set is defined as the T_{gq}-intersection of the fuzzified inequalities. For its membership function one can obtain

$$\vartheta_C(\mathbf{x}) = g^{(-1)}(\|(\mathcal{A}\mathbf{x} - \boldsymbol{\alpha}_0)_+\|_q / D_0(\mathbf{x})), \tag{6}$$

where $(\mathcal{A}\mathbf{x} - \boldsymbol{\alpha}_0)_+$ denotes the vector in which the i-th coordinate is defined by $\max(0, \sum_{j=1}^{n} \alpha_{ij} x_j - \alpha_{i0})$ for every $i = 1, \dots, s$.

Introducing the notions $\vartheta^* = \sup_{\mathbf{x} \in \mathbb{R}^n} \vartheta_C(\mathbf{x})$ and $C_\vartheta^* = \{\mathbf{x} \in \mathbb{R}^n : \vartheta(\mathbf{x}) = \vartheta^*\}$ the following statements are valid: if $\vartheta^* = 1$ and $C_\vartheta^* \neq \emptyset$, then (1)–(3) has solution; if $\vartheta^* = 0$, then there is no consistent perturbation of the constraint set (2); if $0 < \vartheta^* < 1$, then there is no solution of (1)–(3) in the classical sense.

The concept of the optimality was introduced as follows: Let us seek the optimal (g,p)-fuzzified function value as an element of $\mathcal{F}_{gd}$. A fuzzy number $\nu_0 = (y,d) \in \mathcal{F}_{gd}$ is a (g,p)-fuzzy aspiration level for the objective function with the optimality rate $\omega(\mathbf{x},y)$, if $\omega(\mathbf{x},y)$ is the (g,p)-fuzzification of the inequality $\ell(\gamma,\mathbf{x}) = \sum_{j=1}^{n} \gamma_j x_j \le y$ by $(\gamma_j,d) \in \mathcal{F}_{gd}$, $j = 1,\dots,n$, $(y,d) \in \mathcal{F}_{gd}$, i.e.

$$\omega(\mathbf{x},y) = g^{(-1)}\Big((\max(0,\ell(\gamma,\mathbf{x}) - y))/D_0(\mathbf{x})\Big) = g^{(-1)}((\ell(\gamma,\mathbf{x}) - y)_+/D_0(\mathbf{x}))$$

The fuzzy optimum set $\omega^*(\mathbf{x},y)$ with the fuzzy aspiration level is the T_{gq} restriction of the optimality rate to the fuzzy feasible set, i.e.

$$\omega^*(\mathbf{x},y) = T_{gq}(\omega(\mathbf{x},y),\vartheta_C(\mathbf{x})) = g^{(-1)}(\Phi(\mathbf{x},y)/D_0(\mathbf{x})),$$

where $\Phi(\mathbf{x},y) = \|((\ell(\gamma,\mathbf{x}) - y)_+, \|(\mathcal{A}\mathbf{x} - \boldsymbol{\alpha}_0)_+\|_q)\|_q$. The minimal $y = y^*$ and the corresponding x^* are such that $\omega^*(\mathbf{x},y)$ would be as great as possible. To reach this aim let us find the minimal root of the equation:

$$\sup_{\mathbf{x}\in\mathbb{R}^n_+} \omega^*(\mathbf{x},y) = \vartheta^*. \tag{7}$$

In the following we will study the relation between this approach and the parametric LP problem, which will be built using the above results.

4 Parametric approach

Let us consider the case when $p = 1$, $(q = \infty)$ and the side function in (4) is linear, i.e. $g : I \to I$, $g(a) = 1 - a$. This function generates the Łukasiewicz t-norm. Since $\|(\mathcal{A}\mathbf{x} - \boldsymbol{\alpha}_0)_+\|_q = \max_{i=1,\dots,s} (\ell(\boldsymbol{\alpha}_i,\mathbf{x}) - \alpha_{i0})_+$ the $(g,1)$-fuzzy feasible set is defined by the membership function

$$\vartheta_C(\mathbf{x}) = \begin{cases} 1 - \max_i((\ell(\boldsymbol{\alpha}_i,\mathbf{x}) - \alpha_{i0})_+/D_0(\mathbf{x})) \\ \qquad\qquad\qquad \text{if } 0 \le \max_i(\ell(\boldsymbol{\alpha}_i,\mathbf{x}) - \alpha_{i0})_+ \le D_0(\mathbf{x}), \\ 0 \qquad\qquad\qquad \text{otherwise}. \end{cases}$$

Let us modify this membership function of the fuzzy feasible set and the fuzzy objective function value as follows:

$$\tilde{\vartheta}_C(\boldsymbol{x}) = \begin{cases} 1 - \max_i((\ell(\boldsymbol{\alpha}_i, \boldsymbol{x}) - \alpha_{i0})_+/d) & \text{if } 0 \le \max_i(\ell(\boldsymbol{\alpha}_i, \boldsymbol{x}) - \alpha_{i0})_+ \le d, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{\ell}_g(\boldsymbol{\gamma}, \boldsymbol{x})(y) = \max\{0, 1 - |y - \ell(\boldsymbol{\gamma}, \boldsymbol{x})|/d\}.$$

The modified fuzzy feasible set can be described more explicitly in the form

$$\tilde{\vartheta}_C(\boldsymbol{x}) = \sup\{\beta : \sum_{j=1}^{n} \alpha_{ij} x_j \le \alpha_{i0} + d(1-\beta),\ x_j \ge 0,\ i = 1, \dots, s,\ \beta \in (0,1]\}.$$

Clearly $\tilde{\vartheta}_C \subset \vartheta_C$ since $D_0(\boldsymbol{x}) \ge d$.

Let us use the optimum concept described in the previous section for the modified fuzzy LP problem.

The λ-cut of the modified $(g,1)$-fuzzy feasible set is:

$$\begin{aligned} [\tilde{\vartheta}_C]_\lambda &= \{\boldsymbol{x} \in \mathbb{R}^n : \tilde{\vartheta}_C(\boldsymbol{x}) \ge \lambda\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{j=1}^{n} \alpha_{ij} x_j \le \alpha_{i0} + d(1-\lambda),\ x_j \ge 0,\ i = 1, \dots, s\}. \end{aligned}$$

With a fixed λ let $\boldsymbol{x}(\lambda)$ denote the solution of the problem $\mathbf{P}(\lambda)$:

$$\sum_{j=1}^{n} \gamma_j x_j \longrightarrow \min \quad \text{subject to} \quad \boldsymbol{x} \in [\tilde{\vartheta}_C]_\lambda . \tag{8}$$

Then the following result holds:

THEOREM 1. *The optimal value in* (7) *is reached from the parametric programming problem* (8) *by*

$$\boldsymbol{x}^* = \boldsymbol{x}(\theta),\ y^* = \sum_{j=1}^{n} \gamma_j x_j^* - (1-\theta)d,$$

where

$$\theta = \sup_{\lambda \in (0,1]} \{\lambda : \boldsymbol{P}(\lambda) \text{ has optimal solution}\}. \tag{9}$$

Moreover, $\tilde{\vartheta}^*_C = \theta$.

Proof. If x^* is a point where $\tilde{\omega}^*(x, y)$ reaches the maximum value, then $\tilde{\vartheta}_C(x^*) = \tilde{\vartheta}^*$ and $\tilde{\omega}(x^*, y) \geq \tilde{\vartheta}^*$. Since $\tilde{\vartheta}^* = \tilde{\vartheta}_C(x^*) \leq \tilde{\vartheta}_C(x(\theta)) = \theta$, we obtain that $\tilde{\vartheta}^* = \theta$ and the minimal y, for which $\tilde{\omega}(x, y) \geq \tilde{\vartheta}^*$ fulfills, is $y^* = \sum_{j=1}^{n} \gamma_j x_j^* - (1-\theta)d$. If x^* is not optimal for $P(\theta)$, then there exists $y(\theta) = y^* - (\sum_{j=1}^{n} \gamma_j x_j^* - \sum_{j=1}^{n} \gamma_j x_j(\theta))$ belonging to $x(\theta)$ such that $y(\theta) < y^*$ and the pair $(x(\theta), y(\theta))$ satisfies (7). This contradicts to the minimality of y^*. □

When the generator function $g' : [0,1] \to [0, g_0]$, $g'(1) = 0$, $g'(0) = g_0$ is not linear, then it is easy to show that there exists a strictly increasing function $r : [0,1] \to [0, g_0]$, namely $r = g' \circ g^{-1}$, such that $g' = r \circ g$. From the strictly monotonicity of r and the statement of the Theorem 1 it follows that (7) has the same solution (x^*, y^*) independently from the choice of the generator function. Moreover, if ϑ^*_g and $\vartheta^*_{g'}$ denote the optimal membership value belonging to the problems using generator functions g and g', respectively, one can prove easily that

$$\vartheta^*_{g'} = g^{-1}\Big(r^{-1}\big(g(\vartheta^*_g)\big)\Big). \tag{10}$$

(10) means that we can obtain the different degrees for the optimal solution using different nonlinear generator functions. Clearly, if $\vartheta^*_g = 1$, then $\vartheta^*_{g'} = 1$ for all generator functions.

5 Numerical example

Let us consider the problem

$$x_1 + x_2 \longrightarrow \min$$

subject to

$$-x_1 - 3x_2 \leq -9$$

$$2x_1 + x_2 \leq 2$$

$$-4x_1 - 3x_2 \leq -17$$

$$x_j \geq 0$$

and let $d = 5$ be the tolerate margin. The solution of the associated parametrical programming problem is

$$x(\beta) = \begin{cases} (2.666, 0.444 + 1.66\beta & \text{for } \beta \in (0, 0.1838\,], \\ (4.5 - 10\beta, 15\beta - 2) & \text{for} \beta \in (0.1838, 0.45\,]. \end{cases}$$

If we use different generator functions we obtain:

$g(x)$	$r(t)$	ϑ^*	x^*	y^*
$1-x$	t	0.45	$(0, 4.75)$	$4.75 - 5 \cdot 0.55 = 2.0$
$(1-x)^p$	t^p	$1 - 0.55^{1/p}$	$(0, 4.75)$	$4.75 - 5 \cdot 0.55^{1/p}$
$(1-x)^2$	t^2	0.2584	$(0, 4.75)$	1.042
$-\ln x$	$-\ln(1-t)$	0.577	$(0, 4.75)$	2.635

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