Induced Ordered Weighted Geometric Operators and Their Use in the Aggregation of Multiplicative Preference Relations

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In this article, we introduce the induced ordered weighted geometric (IOWG) operator and its properties. This is a more general type of OWG operator, which is based on the induced ordered weighted averaging (IOWA) operator. We provide some IOWG operators to aggregate multiplicative preference relations in group decision-making (GDM) problems. In particular, we present the importance IOWG (I-IOWG) operator, which induces the ordering of the argument values based on the importance of the information sources; the consistency IOWG (C-IOWG) operator, which induces the ordering of the argument values based on the consistency of the information sources; and the preference IOWG (P-IOWG) operator, which induces the ordering of the argument values based on the relative preference values associated with each one of them. We also provide a procedure to deal with "ties" regarding the ordering induced by the application of one of these IOWG operators. This procedure consists of a sequential application of the aforementioned IOWG operators. Finally, we analyze the reciprocity and consistency properties of the collective multiplicative preference relations obtained using IOWG operators. © 2004 Wiley Periodicals, Inc.

1. INTRODUCTION

The aggregation of experts' preferences consisting of combining the individual preferences into a collective one in such a way that all of the properties contained in all the individual preferences are summarized or reflected, is a

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INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 19, 233–255 (2004)

Published online in Wiley InterScience (www.interscience.wiley.com). • DOI 10.1002/int.10172

necessary and very important task to perform when we want to obtain a final solution of multicriteria decision-making (MCDM) or group decision-making (GDM) problems.[1–3]

Yager[4] provided a family of averaging operators called the ordered weighted averaging (OWA) operators, which are commutative, idempotent, continuous, monotonic, neutral, compensative, and stable for positive linear transformations.[5] The OWA operators have been implemented extensively in the last few years in the resolution process of different problems (see Ref. [6] and more recent applications.[7–15] They also have proved to be very important in solving GDM problems because they allow the implementation of the *concept of fuzzy majority,* which is fundamental when seeking a final solution of consensus.[16,17]

As shown in Refs. 18 and 19, the proper aggregation operator of ratio-scale measurements is not the *arithmetic mean* but the *geometric mean.* However, this operator does not allow the concept of fuzzy majority to be incorporated in the decision-making processes. The OWA operator behaves in a similar way to the arithmetic mean and, therefore, we can not use it in decision-making processes with ratio-scale measurements. Chiclana et al.[20] introduced the ordered weighted geometric (OWG) operator based on the OWA operator and the geometric mean. The OWG operator allows the implementation of the concept of fuzzy majority in decision-making processes with ratio-scale assessments in a similar way as the OWA operators. Its properties and applications in decision making under multiplicative preference relations are presented in Refs. 7 and 21 and a complete study of its origin and uses in MCDM can be consulted in Ref. 22.

A fundamental aspect of the OWA and OWG operators is the reordering of the arguments to be aggregated, based on the magnitude of their respective values, which allows us to give importance to values in opposition to the weighted mean (WM) operators that compute an aggregated value taking into account the reliability of the sources of information. However, it is clear that a set of values can be reordered in a different way to the one used by the OWA and OWG operators. To do this, a criterion has to be defined to induce a specific ordering of the arguments to be aggregated before a WM operator can be applied. This is the idea on which Yager and Filev based the definition of the induced OWA (IOWA) operator.[23]

Thus, it is the reordering step of the arguments to be aggregated where the difference between the OWA operator and the IOWA operator resides. The OWA operators order the arguments by their value, whereas the IOWA operators induce their ordering by using an additional variable or criterion called the order-inducing variable. In fact, the OWA operator as well as the weighted averaging (WA) operator are included in the more general class of IOWA operators.[24]

Based on this idea and the aforementioned fact that OWA operators are not appropriate aggregation operators of ratio-scale measurements, we introduce the induced ordered weighted geometric (IOWG) operators. These operators allow us to take control of the aggregation stage of any MCDM or GDM problem in the sense that importance can be given to the values to be aggregated as the OWG operators do or to the information sources as the weighted geometric mean (WGM) operators do. We provide some IOWG operators to aggregate multiplicative preference relations in GDM problems with ratio-scale preference assessments. In particular, we present the importance IOWG (I-IOWG) operator, which induces the ordering of the argument values based on the importance of the information sources; the consistency IOWG (C-IOWG) operator, which induces the ordering of the argument values based on the consistency of the information sources; and the preference IOWG (P-IOWG) operator, which induces the ordering of the argument values based on the relative preference values associated to each one of them. We also provide a different procedure to deal with "ties" regarding the ordering induced by the application of one of these IOWG operators. This procedure consists of a conjunction and sequential application of the aforementioned IOWG operators. Finally, we analyze the reciprocity and consistency properties of the collective multiplicative preference relations obtained using IOWG operators in aggregation processes.

To do this, this work is set out as follows. In Section 2, we summarize the basic operators used in this study: the OWA, OWG, and IOWA operators. In Section 3, we present the IOWG operator and its properties. In Section 4, we define three particular cases of IOWG operators used to aggregate multiplicative preference relations: the I-IOWG, C-IOWG, and P-IOWG operators. In Section 4, a procedure is proposed to deal with "ties" that could appear in the ordering induced by the application of one of the foregoing IOWG operators in the aggregation of multiplicative preference relations. This procedure is different from the one proposed by Yager and Filev.[23] In Section 5, we show that, in general, IOWG operators maintain the reciprocity property of multiplicative preference relations as well as the consistency property. Finally, in Section 6 we draw our conclusions.

2. PRELIMINARIES: OWA, OWG, AND IOWA OPERATORS

We start this section by providing the definitions that are needed to justify the introduction of the IOWG operator.

2.1. The OWA Operator

Chiclana et al.'s[25] GDM problems were considered in which the information about the alternatives was represented using *fuzzy preference relations* and a *fuzzy majority-guided choice scheme* was designed, which follows two steps to reach a final decision from the synthesis of performance degrees of the majority of experts: (i) aggregation and (ii) exploitation. This choice scheme is based on the OWA operator.[4]

DEFINITION 1. *An OWA operator of dimension n is a function* $\phi : \mathbb{R}^n \to \mathbb{R}$, *which has associated a set of weights or weighting vector* $\mathbf{W} = (w_1, \ldots, w_n)$ *to it, so that* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$ *and is defined to aggregate a list of values* $\{p_1, \ldots, p_n\}$ *according to the following expression*:

$$\phi(p_1, \ldots, p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *a permutation such that* $p_{\sigma(i)} \geq p_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, *i.e.*, $p_{\sigma(i)}$ *is the ith highest value in the set* $\{p_1, \ldots, p_n\}$.

A normal question in the definition of the OWA operator is how to obtain the associated weighting vector. In Ref. 4, Yager proposed two ways to obtain it. The first approach is to use some kind of learning mechanism using some sample data; the second approach is to try to give some semantics or meaning to the weights. The latter allowed multiple applications on areas of fuzzy and multivalued logic, evidence theory, design of fuzzy controllers, and the quantifier-guided aggregations.

In the case of quantifier-guided aggregations, the OWA operator has been used to implement the concept of fuzzy majority in the aggregation phase by means of the *fuzzy quantifiers,*[26] which are used to calculate its weights, which in the case of a nondecreasing relative quantifier Q is expressed as follows[4]:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \ldots, n$$

When a fuzzy quantifier Q is used to compute the weights of the OWA operator ϕ, then it is symbolized by ϕ_Q.

Example 1. Suppose three experts provide the following fuzzy preference relations on a set of three alternatives

$$\mathbf{P}^1 = \begin{pmatrix} 0.5 & 0.75 & 0.87 \\ 0.25 & 0.5 & 0.66 \\ 0.13 & 0.34 & 0.5 \end{pmatrix}; \quad \mathbf{P}^2 = \begin{pmatrix} 0.5 & 0.66 & 0.94 \\ 0.34 & 0.5 & 0.87 \\ 0.06 & 0.13 & 0.5 \end{pmatrix}; \quad \mathbf{P}^3 = \begin{pmatrix} 0.5 & 0.66 & 0.75 \\ 0.34 & 0.5 & 0.66 \\ 0.25 & 0.34 & 0.5 \end{pmatrix}$$

Following the choice scheme defined in Ref. 25, if we aggregate them by using an OWA operator guided by the fuzzy linguistic quantifier "most of," i.e., using its corresponding weighting vector (1/15, 10/15, 4/15), then we have the following collective preference relation:

$$\mathbf{P}^c = \phi_{\text{most}}(\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3) = \begin{pmatrix} 0.5 & 0.67 & 0.84 \\ 0.32 & 0.5 & 0.67 \\ 0.12 & 0.28 & 0.5 \end{pmatrix}$$

in which its elements can be interpreted as the preference degree of one alternative over another for most of the experts.

2.2. The OWG Operator

The decision problem when the experts express their preferences using multiplicative preference relations has been solved by Saaty using the decision analytic hierarchical process (AHP), which obtains the set of solution alternatives by means of the eigenvector method.[27] However, this decision process is not guided by the concept of fuzzy majority. Chiclana et al.[7] obtained the transformation function between multiplicative and fuzzy preference relations, which is given in the following result.

PROPOSITION 1. *Suppose that we have a set of alternatives* $X = \{x_1, \ldots, x_n\}$ *and associated with it, a multiplicative reciprocal preference relation* $\mathbf{A} = (a_{ij})$, *with* $a_{ij} \in [1/9, 9]$ *and* $a_{ij} \cdot a_{ji} = 1$, $\forall i, j$. *Then the corresponding fuzzy reciprocal preference relation* $\mathbf{P} = (p_{ij})$, *associated with* $\mathbf{A}$, *with* $p_{ij} \in [0, 1]$ *and* $p_{ij} + p_{ji} = 1$, $\forall i, j$ *is given as follows*: $p_{ij} = f(a_{ij}) = \frac{1}{2}(1 + log_9 a_{ij})$.

The foregoing transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. Based on this, Chiclana et al.[7,21] considered GDM problems where the information about the alternatives is represented using *multiplicative preference relations* and designed a *fuzzy majority–guided choice scheme* based on the quantifier-guided OWG operator.[20,22]

DEFINITION 2. *An OWG operator of dimension n is a function* $\phi^G : \mathbb{R}^n \rightarrow \mathbb{R}$, *to which a set of weights or weighting vectors is associated*, $\mathbf{W} = (w_1, \ldots, w_n)$, *such that* $w_i \in [0, 1]$ *and* $\sum_i w_i = 1$, *and it is defined to aggregate a list of values* $\{a_1, \ldots, a_n\}$ *according to the following expression*:

$$\phi^G(a_1, \ldots, a_n) = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i}$$

where $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *is a permutation such that* $a_{\sigma(i)} \geqslant a_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, *i.e.*, $a_{\sigma(i)}$ *is the ith highest value in the set* $\{a_1, \ldots, a_n\}$.

The OWG operators are continuous, compensative, commutative, and idempotent and are comprised between the maximum and the minimum.[20,22,28]

Because the OWG operator is based on the OWA operator, it is clear that the weighting vector $\mathbf{W}$ can be obtained by the same method used in the case of the OWA operator, i.e., the vector may be obtained using a fuzzy quantifier Q representing the concept of fuzzy majority. When a fuzzy quantifier Q is used to compute the weights of the OWG operator ϕ^G, this is symbolized by ϕ^G_Q. In this way, the concept of fuzzy majority can be implemented in the decision process by using appropriate fuzzy linguistic quantifiers to calculate the weighting vectors of the OWG operator to be used in the aggregation stages of the decision process.

Example 2. Suppose a set of three experts provide the following multiplicative preference relations on a set of three alternatives:

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 3 & 5 \\ 1/3 & 1 & 2 \\ 1/5 & 1/2 & 1 \end{pmatrix}; \quad \mathbf{A}^2 = \begin{pmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{pmatrix}; \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$$

Following the choice scheme defined in Refs. 20 and 22, the collective multiplicative preference relation obtained by using an OWG operator guided by the same linguistic quantifier "most of" is

$$\mathbf{A}^c = \phi^G_{\text{most}}(\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3) = \begin{pmatrix} 1 & 1.48 & 3.21 \\ 0.35 & 1 & 2.17 \\ 0.41 & 0.28 & 1 \end{pmatrix}$$

in which its elements can be interpreted as the preference intensity measured in [1/9, 9][27] of one alternative over another for most of the experts.

2.3. The IOWA Operator

Yager and Filev[23] introduced a more general type of OWA operator, which they named the IOWA operator.

Definition 3. *An IOWA operator of dimension n is a function* $\Phi_{\mathbf{W}} : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$, *to which a set of weights or weighting vector is associated*, $\mathbf{W} = (w_1, \ldots, w_n)$, *such that* $w_i \in [0, 1]$ *and* $\Sigma_i\, w_i = 1$, *and it is defined to aggregate the set of second arguments of a list of n two tuples* $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ *according to the following expression*:

$$\Phi_{\mathbf{W}}(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

with $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *being a permutation such that* $u_{\sigma(i)} \geqslant u_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, *i.e.* $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ *is the two tuple with* $u_{\sigma(i)}$ *the ith highest value in the set* $\{u_1, \ldots, u_n\}$.

In the foregoing definition, the reordering of the set of values to be aggregated, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated with them, which is based on their magnitude. Because of this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order-inducing variable and $\{p_1, \ldots, p_n\}$ the values of the argument variable.[23,24] As we have already mentioned, the main difference between the OWA operator and the IOWA operator resides in the reordering step of the argument variable. In the case of the OWA operator, this reordering is based on the magnitude of the values to be aggregated, and in the case of IOWA operator, an order-inducing variable has to be defined as the criterion to induce that reordering.

An immediate consequence of this definition is that when the order-inducing variable is the argument variable, the IOWA operator is reduced to the OWA operator. For a detailed list of properties and uses of the IOWA operators, the reader should consult Refs. 23, 24 and 29–32.

Remark 1. In this study, we will focus on the aggregation of numerical preferences, which is why we are assuming that the nature of the first argument of the IOWA operators also is numeric, although it could be of a linguistic nature.[23,24,30,32]

Example 3. If we again take the same three experts as in Example 1 and if each expert had a value associated with them, then it is clear that these values could be used as the order-inducing values throughout the aggregation process, and we could build an IOWA operator based on these values.

Suppose that the values associated with the three experts are $\mathbf{b} = (0.65, 0.13, 0.22)$. If we use these values to induce the ordering of the fuzzy preference values to be aggregated and the same fuzzy linguistic quantifier "most of," then we obtain the following collective fuzzy preference relation:

$$\mathbf{P}^c = \Phi_{\text{most}}(\langle 0.65, \mathbf{P}^1\rangle, \langle 0.13, \mathbf{P}^2\rangle, \langle 0.22, \mathbf{P}^3\rangle) = \begin{pmatrix} 0.5 & 0.67 & 0.81 \\ 0.33 & 0.5 & 0.72 \\ 0.19 & 0.28 & 0.5 \end{pmatrix}$$

For example, the values p_{13}^c and p_{31}^c are obtained as follows:

$$p_{13}^c = \Phi_{\text{most}}(\langle 0.65, 0.87\rangle, \langle 0.13, 0.94\rangle, \langle 0.22, 0.75\rangle)$$
$$= \frac{1}{15}\cdot 0.87 + \frac{10}{15}\cdot 0.75 + \frac{4}{15}\cdot 0.94 = \frac{12.13}{15} \simeq 0.81$$

$$p_{31}^c = \Phi_{\text{most}}(\langle 0.65, 0.13\rangle, \langle 0.13, 0.06\rangle, \langle 0.22, 0.25\rangle)$$
$$= \frac{1}{15}\cdot 0.13 + \frac{10}{15}\cdot 0.25 + \frac{4}{15}\cdot 0.06 = \frac{2.87}{15} \simeq 0.19$$

Remark 2. The results obtained using this type of induced ordered aggregation are different from the ones obtained in Example 1, where we used the simple OWA operator. Furthermore, in this case, we observe that the collective preference relation verifies the additive reciprocity property $p_{ij}^c + p_{ji}^c = 1, \quad \forall i, j$.

3. THE IOWG OPERATOR AND ITS PROPERTIES

In this section, we introduce the IOWG operator and study its properties.

3.1. IOWG Operator

Suppose that we want to aggregate a set of two tuples $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$, where $\{u_1, \ldots, u_n\}$ is the set of order-inducing values associated with the set of argument values $\{a_1, \ldots, a_n\}$ given on the basis of a positive ratio scale. We can use the IOWA operator not on the set of two-tuple $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$ but on the set $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$, where the argument values $\{p_1, \ldots, p_n\}$ are obtained using the foregoing transformation function f of Proposition 1, i.e., $p_i = f(a_i) = \frac{1}{2}(1 + \log_9 a_i)$. Thus, we obtain

$$p = \Phi_{\mathbf{W}}(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

where $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the two tuple with $u_{\sigma(i)}$ the i-th highest value in the set $\{u_1, \ldots, u_n\}$.

The set of two tuples $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$ and $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ have the same set of order-inducing values, which implies that the induced orderings of the arguments $\{a_1, \ldots, a_n\}$ and $\{p_1, \ldots, p_n\}$ are the same, and, hence,

$$p = \sum_{i=1}^{n} w_i \cdot \frac{1}{2}\,(1 + \log_9 a_{\sigma(i)}) = \frac{1}{2}\left(\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i \cdot \log_9 a_{\sigma(i)}\right)$$

$$= \frac{1}{2}\left(1 + \sum_{i=1}^{n} \log_9 (a_{\sigma(i)})^{w_i}\right) = \frac{1}{2}\left(1 + \log_9 \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i}\right)$$

This last expression justifies the definition of the IOWG operator as follows:

DEFINITION 4. *An IOWG operator of dimension n is a function* $\Phi^G_{\mathbf{W}} : (\mathbb{R} \times \mathbb{R}^+)^n \to \mathbb{R}^+$, *to which a set of weights or a weighting vector is associated*, $\mathbf{W} = (w_1, \ldots, w_n)$, *such that* $w_i \in [0, 1]$ *and* $\Sigma_i\, w_i = 1$, *and it is defined to aggregate the set of second arguments of a list of n two-tuples* $\{\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\}$, *given on the basis of a positive ratio scale*, *according to the following expression*:

$$\Phi^G_{\mathbf{W}}(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i}$$

being $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, *a permutation such that* $u_{\sigma(i)} \geqslant u_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, *i.e.*, $\langle u_{\sigma(i)}, a_{\sigma(i)}\rangle$ *is the two tuple with* $u_{\sigma(i)}$ *the ith highest value in the set* $\{u_1, \ldots, u_n\}$.

Example 4. If we again take the same three experts as in Example 2 that have the following values $\mathbf{b} = (0.65, 0.13, 0.22)$ associated to them, the collective multiplicative preference relation obtained using these values to induce the ordering and the same fuzzy linguistic quantifier "most of" is

$$\mathbf{A}^c = \Phi^G_{\text{most}}(\langle 0.65, \mathbf{A}^1\rangle, \langle 0.13, \mathbf{A}^2\rangle, \langle 0.22, \mathbf{A}^3\rangle) = \begin{pmatrix} 1 & 1.08 & 3.89 \\ 1/1.08 & 1 & 2.55 \\ 1/3.89 & 1/2.55 & 1 \end{pmatrix}$$

For example, the values a^c_{12} and a^c_{21} are obtained as follows:

$$a^c_{12} = \Phi^G_{\text{most}}(\langle 0.65, 3\rangle, \langle 0.13, 2\rangle, \langle 0.22, 2\rangle) = 3^{1/15} \cdot 2^{10/15} \cdot 2^{4/15} = \sqrt[15]{3 \cdot 2^{14}} \simeq 1.08$$

$$a^c_{21} = \Phi^G_{\text{most}}\left(\left\langle 0.65, \frac{1}{3}\right\rangle, \left\langle 0.13, \frac{1}{2}\right\rangle, \left\langle 0.22, \frac{1}{2}\right\rangle\right)$$

$$= \left(\frac{1}{3}\right)^{1/15} \cdot \left(\frac{1}{2}\right)^{10/15} \cdot \left(\frac{1}{2}\right)^{4/15} = \frac{1}{\sqrt[15]{3 \cdot 2^{14}}} \simeq \frac{1}{1.08}$$

Remark 3. The collective multiplicative preference values obtained here are different from the ones obtained by applying the usual OWG operator, and, again, the multiplicative reciprocal property $a_{ij}^c \cdot a_{ji}^c = 1$, $\forall i, j$ is verified.

3.2. Properties of the IOWG Operators

In this section, we look at the properties associated with these IOWG operators $\Phi_{\mathbf{W}}^G$.

3.2.1. IOWG Operators Are Commutative

Suppose that $(\langle u_{\sigma_1(1)}, a_{\sigma_1(1)}\rangle, \ldots, \langle u_{\sigma_1(n)}, a_{\sigma_1(n)}\rangle)$ is a reordering of the set of two tuples $(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle)$, σ_1 being a permutation of the set $\{1, \ldots, n\}$. We have to prove that

$$\Phi_{\mathbf{W}}^G(\langle u_{\sigma_1(1)}, a_{\sigma_1(1)}\rangle, \ldots, \langle u_{\sigma_1(n)}, a_{\sigma_1(n)}\rangle) = \Phi_{\mathbf{W}}^G(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle)$$

Because the sets of order-inducing values $(u_1, \ldots, u_n)$ and $(u_{\sigma_1(1)}, \ldots, u_{\sigma_1(n)})$ have the same elements, the ordering of them from highest to lowest is unique, and, thus, $\{u_{\sigma(1)}, \ldots, u_{\sigma(n)}\}$ and $\{u_{\sigma_2(\sigma_1(1))}, \ldots, u_{\sigma_2(\sigma_1(n))}\}$, σ and σ_2 being two permutations such that $u_{\sigma(i)} \geqslant u_{\sigma(i)}$ and $u_{\sigma_2(\sigma_1(i))} \geqslant u_{\sigma_2(\sigma_1(i+1))}$, $\forall i = 1, \ldots, n-1$, respectively, are the same sets, i.e., $\sigma \equiv \sigma_2 \circ \sigma_1$ and, hence,

$$\Phi_{\mathbf{W}}^G(\langle u_{\sigma_1(1)}, a_{\sigma_1(1)}\rangle, \ldots, \langle u_{\sigma_1(n)}, a_{\sigma_1(n)}\rangle) = \prod_{i=1}^{n} (a_{\sigma_2(\sigma_1(i))})^{w_i}$$

$$= \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} = \Phi_{\mathbf{W}}^G(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle)$$

3.2.2. IOWG Operators Are or/and Operators; i.e., They Are Located Between the Minimum and the Maximum of the Arguments to be Aggregated

Suppose that we want to aggregate the set of two tuples $(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle)$ and that σ is the permutation such that $u_{\sigma(i)}$ the ith highest value in the set $\{u_1, \ldots, u_n\}$. If we denote $a_m = \min_i\{a_i\}$ and $a_M = \max_i\{a_i\}$, then as $a_i \geqslant 0$, $\forall i \in \{1, \ldots, n\}$ we have

$$0 \leqslant a_m \leqslant a_{\sigma(i)} \leqslant a_M, \quad \forall i \in \{1, \ldots, n\}$$

Applying the elemental properties

$$0 \leqslant a \leqslant b \Rightarrow 0 \leqslant a^c \leqslant b^c \ \forall c \geqslant 0 \quad \text{and}$$

$$0 \leqslant a \leqslant b \wedge 0 \leqslant c \leqslant d \Rightarrow 0 \leqslant a \cdot c \leqslant b \cdot d$$

we have

$$0 \leqslant (a_m)^{w_i} \leqslant (a_{\sigma(i)})^{w_i} \leqslant (a_M)^{w_i}, \quad \forall i \in \{1, \ldots, n\}, \quad w_i \geqslant 0, \quad \sum_{i=1}^{n} w_i = 1$$

and

$$0 \leqslant \prod_{i=1}^{n} (a_m)^{w_i} \leqslant \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} \leqslant \prod_{i=1}^{n} (a_M)^{w_i}, \quad \forall w_i \geqslant 0, \quad \sum_{i=1}^{n} w_i = 1$$

which implies

$$\min_i \{a_i\} \leqslant \Phi_{\mathbf{W}}^{G}(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) \leqslant \max_i \{a_i\}$$

3.2.3. *IOWG Operators Are Idempotent with Respect to the Argument Variable*

In fact, this is a consequence of the foregoing property, because if $a_i = a \ \forall i$, then $\min_i\{a_i\} = \max_i\{a_i\} = a$ and, therefore, $\Phi_{\mathbf{W}}^{G}(\langle u_1, a\rangle, \ldots, \langle u_n, a\rangle) = a$.

3.2.4. *IOWG Operators Are Increasingly Monotonous with Respect to the Argument Variable*

Suppose two sets of two tuples $(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle)$ and $(\langle u_1, b_1\rangle, \ldots, \langle u_n, b_n\rangle)$ such that $a_i \leqslant b_i \ \forall i$. Because the order-inducing values are the same for both sets, then $a_{\sigma(i)} \leqslant b_{\sigma(i)} \ \forall i$, and, hence,

$$\Phi_{\mathbf{W}}^{G}(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) \leqslant \Phi_{\mathbf{W}}^{G}(\langle u_1, b_1\rangle, \ldots, \langle u_n, b_n\rangle)$$

The proof of the monotonicity property is based on the assumption that the order-inducing values are unchanged. If this is not the case, then monotonicity does not necessarily hold.

3.2.5. *The IOWG Operator Is Reduced to the Geometric Mean (GM) Operator When $w_i = 1/n, \ \forall i$*

In fact, if $\mathbf{W} = (1/n, \ldots, 1/n)$, we have

$$\begin{aligned}\Phi_{\mathbf{W}}^{G}(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) &= \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} \\ &= \prod_{i=1}^{n} (a_{\sigma(i)})^{1/n} = \prod_{i=1}^{n} (a_i)^{1/n} = \mathrm{GM}(a_1, \ldots, a_n)\end{aligned}$$

3.2.6. *The IOWG Operator Is Reduced to the WGM Operator When the Two Tuples Have the Following Expression $\langle f(n - i + 1), a_i \rangle$, f Being an Increasing Function*

If $u_i = f(n - i + 1)$ and f is an increasing function, then $\sigma(i) = i$; thus,

$$\Phi^G_{\mathbf{W}}(\langle f(n), a_1\rangle, \ldots, \langle f(1), a_n\rangle) = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} = \prod_{i=1}^{n} (a_i)^{w_i} = \mathrm{WGM}(a_1, \ldots, a_n)$$

3.2.7. *The IOWG Operator Is Reduced to the OWG Operator When the Two Tuples Have the Following Expression $\langle f(a_i), a_i \rangle$, f Being an Increasing Function*

If $u_i = f(a_i)$, f being an increasing function, then the ith highest value of $(u_1, \ldots, u_n)$ is $u_{\sigma(i)}$ if and only if $a_{\sigma(i)}$ is the ith highest value of $(a_1, \ldots, a_n)$, i.e.,

$$\Phi^G_{\mathbf{W}}(\langle f(a_1), a_1\rangle, \ldots, \langle f(a_n), a_n\rangle) = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} = \phi^G_{\mathbf{W}}(a_1, \ldots, a_n)$$

Additionally, we have the two following cases:

- If $\mathbf{W}^* = (1, 0, \ldots, 0)$, the IOWG operator is reduced to the maximum operator.

$$\Phi^G_{\mathbf{W}^*}(\langle f(a_1), a_1\rangle, \ldots, \langle f(a_n), a_n\rangle) = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} = a_{\sigma(1)} = \max_i\{a_i\}$$

- If $\mathbf{W}_* = (0, \ldots, 0, 1)$, the IOWG operator is reduced to the minimum operator.

$$\Phi^G_{\mathbf{W}_*}(\langle f(a_1), a_1\rangle, \ldots, \langle f(a_n), a_n\rangle) = \prod_{i=1}^{n} (a_{\sigma(i)})^{w_i} = a_{\sigma(n)} = \min_i\{a_i\}$$

4. AGGREGATION OF MULTIPLICATIVE PREFERENCE RELATIONS BY MEANS OF IOWG OPERATORS

The result obtained in the aggregation using a WGM operator summarizes the aggregated values of the information sources taking into account the reliability of these sources. However, the OWG operator combines the information giving weights to the values in relation to their ordering position, diminishing the importance of extreme values by increasing the importance of central ones. Because both the WGM and the OWG operators are special types of IOWG operator, by using an IOWG operator, the type of aggregation method to be implemented can be chosen.

In this section, we present three special cases of IOWG operators for GDM with multiplicative preference relations (i.e., ratio-scale preference assessments). These IOWG operators allow the introduction of some semantics or meaning in the aggregation, and, therefore, allow for better control over the aggregation stage developed in the resolution process. The first two act as the WGM operator because the aggregation is based on the reliability of the information sources, and the third case acts as the OWG operator because the ordering of the argument values is based on a relative magnitude associated with each one of them.

We will suppose that we have a group of experts $E = \{e_1, \ldots, e_m\}$, which provides preferences about a set of alternatives $X = \{x_1, \ldots, x_n\}$ by means of multiplicative preference relations $\{\mathbf{A}^1, \ldots, \mathbf{A}^m\}$, which are reciprocal $a_{ij}^k \cdot a_{ji}^k = 1, \forall i, j, k$.[27]

4.1. The Importance IOWG Operator

In many cases, each expert $e_k \in E$ has an *importance degree* associated with them. This importance degree can be interpreted as a fuzzy subset, $\mu_{\mathrm{I}}: E \to [0, 1]$, in such a way that $\mu_{\mathrm{I}}(e_k) \in [0, 1]$ denotes the importance degree of the opinion provided by the expert e_k. When this is the case, we call this a heterogeneous GDM problem.[33,34]

Assuming that in our context each value $\mu_{\mathrm{I}}(e_k)$ is a weight indicating the importance of the expert e_k, the general procedure for its inclusion in the aggregation process involves the transformation of the preference value a_{ij}^k under the importance degree to generate a new value $\bar{a}_{ij}^k$. This activity is performed by means of a transformation function

$$\bar{a}_{ij}^k = g(a_{ij}^k, \mu_{\mathrm{I}}(e_k))$$

Examples of functions g used in these cases include the minimum operator,[35] the exponential function $g(x, y) = x^y$,[36] or, generally, a t-norm operator.[37]

In our case, we can implement this importance degree variable in our GDM problem as the order-inducing variable to induce the ordering of the argument values before their aggregation. We call this importance degree–based IOWG operator the importance IOWG (I-IOWG) operator and denote it as $\Phi_{\mathbf{W}}^{I-G}$.

DEFINITION 5. *If a set of experts (or criteria) $E = \{e_1, \ldots, e_m\}$ provides preferences about a set of alternatives $X = \{x_1, \ldots, x_n\}$ by means of multiplicative preference relations $\{\mathbf{A}^1, \ldots, \mathbf{A}^m\}$, and each expert has an importance degree $\mu_{\mathrm{I}}(e_k) \in [0, 1]$ associated with them, then an I-IOWG operator of dimension n $\Phi_{\mathbf{W}}^{I-G}$ is an IOWG operator in which its order-inducing values is the set of importance degrees.*

Example 5. Suppose that we have a set of three experts $E = \{e_1, e_2, e_3\}$ and a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$, and suppose that the importance pairwise comparisons of these three experts are given in the following reciprocal multiplicative preference relation

$$\mathbf{I} = \begin{pmatrix} 1 & 6 & 4 \\ 1/6 & 1 & 3 \\ 1/4 & 1/3 & 2 \end{pmatrix}$$

These numbers have the following meaning:

1	Equally important
3	Weakly more important
5	Strongly more important
7	Demonstrably or very strongly more important
9	Absolutely more important
2, 4, 6, and 8	Compromise between slightly differing judgments

According to Saaty, the next step would be the computation of a vector of priorities, in our case of importance, from the given matrix, for which the principal eigenvector is computed and normalized. The vector of importance for this matrix is given by $\mathbf{I} = (0.701, 0.193, 0.106)$.

Suppose that these experts provide the following reciprocal multiplicative preference relations on the set of alternatives:

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 6 & 6 & 3 \\ 1/6 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/3 & 1/3 & 2 & 1 \end{pmatrix}; \quad \mathbf{A}^2 = \begin{pmatrix} 1 & 6 & 6 & 8 \\ 1/6 & 1 & 2 & 3 \\ 1/6 & 1/2 & 1 & 1/2 \\ 1/8 & 1/3 & 2 & 1 \end{pmatrix}; \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 1/5 & 1/3 & 1 \\ 5 & 1 & 4 & 1/5 \\ 3 & 1/4 & 1 & 1/4 \\ 1 & 1/5 & 4 & 1 \end{pmatrix}$$

Using the fuzzy linguistic quantifier "most of," the collective multiplicative preference relation that we obtain is

$$\mathbf{A}^c = \Phi^{I-G}_{\text{most}}(\langle 0.701, \mathbf{A}^1\rangle, \langle 0.193, \mathbf{A}^2\rangle, \langle 0.106, \mathbf{A}^3\rangle) = \begin{pmatrix} 1 & 2.42 & 3.65 & 4.3 \\ 0.41 & 1 & 2.52 & 1.46 \\ 0.27 & 0.4 & 1 & 0.42 \\ 0.23 & 0.68 & 2.38 & 1 \end{pmatrix}$$

in which its elements can be considered as the preference of one alternative over another for most of the more important experts.

4.2. The Consistency IOWG Operator

When the experts have equal importance, i.e., in a homogeneous GDM problem, the application of the I-IOWG operator is reduced to the GM operator. Thus, in this case, the application of the I-IOWG operator does not introduce any new meaning and its application is not advisable. However, in a homogeneous situation, each expert always can have a consistency index (CI) value associated with them. This CI value responds to its multiplicative preference relation and will be used as a substitute for the previous importance degree value. Saaty[27] defined the CI value of a multiplicative preference relation as

$$\text{CI}^k = \frac{\lambda^k_{\max} - n}{n - 1}$$

where $\lambda^k_{\max}$ is the maximum or principal eigenvalue of $\mathbf{A}^k$.

The closer CI^k is to 0 the more consistent is the information provided by the expert e^k, and, thus, more importance should be placed on that information. In other words, we could use these values to define the ordering of the argument values to be aggregated, in which case we would be implementing the concept of consistency in the aggregation process of our decision making. This kind of aggregation process defines an IOWG operator that we call consistency IOWG (C-IOWG) operator and it is denoted as $\Phi_{\mathbf{W}}^{C-G}$.

DEFINITION 6. *If a set of experts $E = \{e_1, \ldots, e_m\}$ provides preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$ by means of multiplicative preference relations $\{\mathbf{A}^1, \ldots, \mathbf{A}^m\}$, then a C-IOWG operator of dimension n $\Phi_{\mathbf{W}}^{C-G}$ is an IOWG operator in which its set of order-inducing values is the set of CI values $\{-\mathrm{CI}^1, \ldots, -\mathrm{CI}^m\}$ associated with the set of experts.*

Example 6. If we take the same data as in Example 2, the CIs associated with these experts are $\mathbf{CI} = (0.002, 0.007, 0.005)$, and the collective multiplicative preference relation obtained by using a C-IOWG operator guided by the same linguistic quantifier "most of" is

$$\mathbf{A}^c = \Phi_{\text{most}}^{C-G}(\langle -0.002, \mathbf{A}^1\rangle, \langle -0.007, \mathbf{A}^2\rangle, \langle -0.005, \mathbf{A}^3\rangle) = \begin{pmatrix} 1 & 1.08 & 3.89 \\ 0.93 & 1 & 2.55 \\ 0.26 & 0.39 & 1 \end{pmatrix}$$

in which its elements can be interpreted as the preference intensity, measured in [1/9, 9],[27] of one alternative over another for most of the more consistent experts.

4.3. The Preference IOWG Operator

When a preference pairwise comparison of a set of alternatives $\{x_1, \ldots, x_n\}$ is given in a fuzzy preference relation $\mathbf{P} = (p_{ij})$, the total sum of the elements of each row i, $\bar{p}_i = \sum_r p_{ir}$, can be interpreted as the total preference of the alternative x_i. The resulting value obtained by dividing an element of that row p_{ij} by $\bar{p}_i$, $\bar{p}_{ij} = p_{ij}/\sum_r p_{ir}$ can be interpreted as the relative preference contribution of that particular element to the total preference of the alternative x_i. In this case, the total sum of all the relative preferences of each row i is 1.

In the case of a multiplicative preference relation $\mathbf{A} = (a_{ij})$, a value can be associated with each element of a row in a way that the product of all of them equals 1. To do this, for each row, the GM of its elements $\bar{a}_i = \sqrt[n]{\prod_r a_{ir}}$ and for each element of that row, its ratio to this GM value $\bar{a}_i$, $\bar{a}_{ij} = a_{ij}/\bar{a}_i = a_{ij}/\sqrt[n]{\prod_r a_{ir}}$, are calculated. The values $\bar{a}_i$ and $\bar{a}_{ij}$ can be interpreted as the multiplicative preference of the alternative x_i and the multiplicative relative preference contribution of the element a_{ij} to the total multiplicative preference of the alternative x_i.

These multiplicative relative preference values can be used as the order-inducing values of an IOWG operator to aggregate a set of multiplicative preference relations. We call this a preference IOWG (P-IOWG) operator and denote it as $\Phi_{\mathbf{W}}^{\mathrm{P}-G}$.

DEFINITION 7. *If a set of experts (or criteria) $E = \{e_1, \ldots, e_m\}$ provides preferences about a set of alternatives $X = \{x_1, \ldots, x_n\}$ by means of multiplicative* *preference relations* $\{\mathbf{A}^1, \ldots, \mathbf{A}^m\}$, *then a P-IOWG operator of dimension* n $\Phi_{\mathbf{W}}^{P-G}$ *is an IOWG operator in which its set of order-inducing values is the set of multiplicative relative preference matrices* $\{\bar{\mathbf{A}}^k = (\bar{a}_{ij}^k);\ k = 1, \ldots, m\}$.

The general expression of this type of P-IOWG operator is the following:

$$\Phi_{\mathbf{W}}^{P-G}(\langle u_{ij}^1, a_{ij}^1\rangle, \ldots, \langle u_{ij}^m, a_{ij}^m\rangle); \quad u_{ij}^k = f_i(a_{i1}^k, \ldots, a_{in}^k)$$

In our case, we use the particular expression $u_{ij}^k = \bar{a}_{ij}^k = f_i(a_{i1}^k, \ldots, a_{in}^k) = a_{ij}^k / \sqrt[n]{\prod_{r=1}^n a_{ir}^k}$.

In the following, we will show that it does not matter which set of the foregoing relative preference values $\{\bar{p}_{ij}^1, \ldots, \bar{p}_{ij}^m\}$, with $\bar{p}_{ij}^k = a_{ij}^k / \sum_{k=1}^n a_{ir}^k$, or $\{\bar{a}_{ij}^1, \ldots, \bar{a}_{ij}^m\}$, with $\bar{a}_{ij}^k = a_{ij}^k / \sqrt[n]{\prod_{r=1}^n a_{ir}^k}$, are used to induce the ordering of the ratio-scale argument values $\{a_{ij}^1, \ldots, a_{ij}^m\}$.

PROPOSITION 2. *If* $\{A_1, \ldots, A_n\}$ *and* $\{B_1, \ldots, B_n\}$ *are two sets of values such that* $\sum_k w_k \cdot A_k \leqslant \sum_k w_k \cdot B_k$, *then* $\sum_k w_k \cdot f(A_k) \leqslant \sum_k w_k \cdot f(B_k)$ *for any function f such that* $f'(x) > 0$, $\forall x$.

Proof. First, any pair of values (A_i, B_i) such that $A_i = B_i$ can be eliminated from both sides of the inequality $\sum_k w_k \cdot A_k \leqslant \sum_k w_k \cdot B_k$. Consequently, we have $(A_i, B_i) \neq (0, 0)\ \forall i$.

If we suppose the contrary, i.e.,

$$\sum_k w_k \cdot f(B_k) < \sum_k w_k \cdot f(A_k)$$

then, we have

$$\sum_k w_k \cdot (f(B_k) - f(A_k)) < 0$$

and, hence,

$$\sum_k w_k \cdot (B_k - A_k) \cdot \frac{f(B_k) - f(A_k)}{B_k - A_k} < 0$$

By the mean theorem value there exists a value C_k such that

$$\frac{f(B_k) - f(A_k)}{B_k - A_k} = f'(C_k)$$

and $\min\{A_k, B_k\} < C_k < \max\{A_k, B_k\}$. Let $f'(C) = \min_k\{f'(C_k)\}$; thus,

$$\begin{aligned} f'(C) \cdot \sum_k w_k \cdot (B_k - A_k) &= \sum_k w_k \cdot (B_k - A_k) \cdot f'(C) \\ &\leqslant \sum_k w_k \cdot (B_k - A_k) \cdot f'(C_k) = \sum_k w_k \cdot (f(B_k) - f(A_k)) < 0 \end{aligned}$$

Because of f differentiable and $f'(C) > 0$; then, $\sum_k w_k \cdot (B_k - A_k) < 0$, which is equivalent to

$$\sum_k w_k \cdot B_k < \sum_k w_k \cdot A_k$$

which contradicts our initial assumption. ∎

A consequence of this property is the following.

COROLLARY 3. *If* $\sum_k w_k \cdot A_k \leqslant \sum_k w_k \cdot B_k$, *then* $\prod_k (A_k)^{w_k} \leqslant \prod_k (B_k)^{w_k}$.

Proof. Taking function $f(x) = \log_9 x$ and applying Proposition 2, we have

$$\sum_k w_k \cdot \log_9 A_k \leqslant \sum_k w_k \cdot \log_9 B_k$$

$$\Updownarrow$$

$$\sum_k \log_9 (A_k)^{w_k} \leqslant \sum_k \log_9 (B_k)^{w_k}$$

$$\Updownarrow$$

$$\log_9 \prod_k (A_k)^{w_k} \leqslant \log_9 \prod_k (B_k)^{w_k}$$

$$\Updownarrow$$

$$\prod_k (A_k)^{w_k} \leqslant \prod_k (B_k)^{w_k}$$ ∎

The following property establishes that the collective preference relations obtained by applying a P-IOWG operator to a set of multiplicative preference ones, using order-inducing values $\bar{a}_{ij}^k = a_{ij}^k / \bar{a}_i^k = a_{ij}^k / \sqrt[n]{\prod_{r=1}^n a_{ir}^k}$ and $\bar{p}_{ij}^k = a_{ij}^k / \bar{p}_i^k = a_{ij}^k / \sum_{r=1}^n a_{ir}^k$ are the same.

COROLLARY 4. *If* $\{A^1, \ldots, A^m\}$ *is a set of multiplicative preference relations*, *then*

$$\Phi_{\mathbf{W}}^{P-G}(\langle \bar{a}_{ij}^1, a_{ij}^1 \rangle, \ldots, \langle \bar{a}_{ij}^m, a_{ij}^m \rangle) = \Phi_{\mathbf{W}}^{P-G}(\langle \bar{p}_{ij}^1, a_{ij}^1 \rangle, \ldots, \langle \bar{p}_{ij}^m, a_{ij}^m \rangle)$$

with

$$\bar{a}_{ij}^k = \frac{a_{ij}^k}{\sqrt[n]{\prod_{r=1}^n a_{ir}^k}} \quad \textit{and} \quad \bar{p}_{ij}^k = \frac{a_{ij}^k}{\sum_{r=1}^n a_{ir}^k}$$

Proof. Suppose that $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that

$$\bar{p}_{ij}^{\sigma(k)} \geqslant \bar{p}_{ij}^{\sigma(k+1)}, \quad \forall k = 1, \ldots, n-1$$

Multiplying both sides of the foregoing inequality by $\bar{p}_i^{\sigma(k)} \cdot \bar{p}_i^{\sigma(k+1)}$, we have

$$a_{ij}^{\sigma(k)} \cdot \bar{p}_i^{\sigma(k+1)} \geqslant a_{ij}^{\sigma(k+1)} \cdot \bar{p}_i^{\sigma(k)}$$

If we denote $C = a_{ij}^{\sigma(k)}$, $D = a_{ij}^{\sigma(k+1)}$, $w_r = \frac{1}{n}$, $A_r = C \cdot a_{ir}^{\sigma(k+1)}$, and $B_r = D \cdot a_{ir}^{\sigma(k)}$, then the foregoing inequality can be rewritten as

$$\sum_{r=1}^{n} w_r \cdot A_r \geqslant \sum_{r=1}^{n} w_r \cdot B_r$$

Applying Corollary 3, we obtain

$$\prod_r (A_r)^{w_r} \geqslant \prod_r (B_r)^{w_r}$$

which is equivalent to

$$\sqrt[n]{\prod_{\mathrm{k}} (\mathrm{C}\cdot a_{\mathrm{ir}}^{\sigma(\mathrm{k}+1)})} \geqslant \sqrt[n]{\prod_{\mathrm{k}} (\mathrm{D}\cdot a_{\mathrm{ir}}^{\sigma(\mathrm{k})})}$$

Rearranging both sides of this inequality we have

$$\frac{C}{\sqrt[n]{\prod_k a_{ir}^{\sigma(k)}}} \geqslant \frac{D}{\sqrt[n]{\prod_k a_{ir}^{\sigma(k+1)}}}$$

i.e.,

$$\bar{a}_{ij}^{\sigma(k)} \geqslant \bar{a}_{ij}^{\sigma(k+1)}, \quad \forall k = 1, \ldots, n-1$$

which means that the orderings of the sets $\{\bar{a}_{ij}^1, \ldots, \bar{a}_{ij}^m\}$ and $\{\bar{p}_{ij}^1, \ldots, \bar{p}_{ij}^m\}$, based on their magnitude, are the same, and, hence,

$$\Phi_{\mathbf{W}}^{P-G}(\langle \bar{a}_{ij}^1, a_{ij}^1\rangle, \ldots, \langle \bar{a}_{ij}^m, a_{ij}^m\rangle) = \Phi_{\mathbf{W}}^{P-G}(\langle \bar{p}_{ij}^1, a_{ij}^1\rangle, \ldots, \langle \bar{p}_{ij}^m, a_{ij}^m\rangle) \qquad \blacksquare$$

The previous IOWG operators, the I-IOWG and the C-IOWG operators, act as the WGM operator because the aggregation is based on the reliability of the information sources, and the P-IOWG operator acts as the OWG operators because the ordering of the argument values is based on a relative magnitude associated with each one of them.

Example 7. Taking the same data as in Example 2,

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 3 & 5 \\ 1/3 & 1 & 2 \\ 1/5 & 1/2 & 1 \end{pmatrix}; \quad \mathbf{A}^2 = \begin{pmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{pmatrix}; \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$$

The multiplicative relative preference matrices $\bar{\mathbf{A}}^k = (\bar{a}_{ij}^k)$ being $\bar{a}_{ij}^k = a_{ij^k}/\sqrt[n]{\prod_r a_{ir}^k}$, associated with these multiplicative preference relations are

$$\bar{\mathbf{A}}^1 = \begin{pmatrix} 0.42 & 1.22 & 2.03 \\ 0.38 & 1.14 & 2.29 \\ 0.43 & 1.08 & 2.15 \end{pmatrix}; \quad \bar{\mathbf{A}}^2 = \begin{pmatrix} 0.41 & 0.83 & 2.9 \\ 0.37 & 0.74 & 3.68 \\ 0.47 & 0.65 & 3.27 \end{pmatrix};$$

$$\bar{\mathbf{A}}^3 = \begin{pmatrix} 0.55 & 1.1 & 1.65 \\ 0.5 & 1 & 2 \\ 0.61 & 0.91 & 1.82 \end{pmatrix}$$

The collective multiplicative preference relation obtained by using the P-IOWG operator guided by the same linguistic quantifier "most of" is

$$\mathbf{A}^c = \Phi_{\text{most}}^{P-G}(\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3) = \begin{pmatrix} 1 & 2.05 & 4.46 \\ 0.61 & 1 & 2.13 \\ 0.19 & 0.39 & 1 \end{pmatrix}$$

The collective preference relation obtained by the application of the P-IOWG operator does not verify the reciprocity property. This is because of the fact that this P-IOWG operator behaves as an OWG operator, which normally does not maintain the reciprocity property.[22]

4.4. A Procedure to Deal with Ties Using IOWG Operators

As we said before, when using IOWG operators, ties could appear and the aggregated values can be different according to the procedure applied, in contrast to the OWG operators, where ties do not affect the aggregated values. In the case of aggregating multiplicative preference relations, when using an IOWG operator, we propose a sequential procedure different from the one proposed by Yager and Filev,[23] which is applied in three steps, as follows:

1. If the GDM problem is heterogeneous, then the I-IOWG operator is applied and if not, the C-IOWG operator is applied.
2. If an I-IOWG operator has been applied in one, then the ordering of the equally important information is induced based on their respective CI values. If a C-IOWG operator has been applied in one, then the ordering of the equally consistent information is induced based on their respective multiplicative relative preference values.
3. Finally, if ties are still present, then their ordering is induced based on their respective magnitude, i.e., applying the usual OWG operator.

Example 8. Suppose a school has to be selected from a set of three $\{A, B, C\}$, for which six independent criteria are used: L, learning; F, friends; S, school life; V, vocational training; P, college preparation; and M, music classes.[27] Suppose that the relative importance degrees of these six criteria are expressed in the following matrix:

$$\mathbf{I} = \begin{pmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 1/4 & 1 & 7 & 3 & 1/5 & 1 \\ 1/3 & 1/7 & 1 & 1/5 & 1/5 & 1/6 \\ 1 & 1/3 & 5 & 1 & 1 & 1/3 \\ 1/3 & 5 & 5 & 1 & 1 & 3 \\ 1/4 & 1 & 6 & 3 & 1/3 & 1 \end{pmatrix}$$

The vector of importance for this matrix is given by $\mathbf{I} = (0.32, 0.14, 0.03, 0.13, 0.24, 0.14)$.

When using these importance values to induce the ordering of the preference values, we observe that there is a tie between the criteria F and M. In this case, we induce their ordering using their respective CI values. The CI values associated with $\mathbf{A}^{\mathrm{F}}$ and $\mathbf{A}^{\mathrm{M}}$ are $\mathrm{CI}^{\mathrm{F}} = 0$ and $\mathrm{CI}^{\mathrm{M}} = 0.025$, respectively, resulting in the following final induced ordering of the six criteria {L, P, F, M, V, S}. We call this aggregation operator the importance consistency–IOWG (IC-OWG) operator.

Suppose that the pairwise comparison matrices of these three schools with respect to this set of six criteria are

$$\mathbf{A}^{\mathrm{L}} = \begin{pmatrix} 1 & 1/3 & 1/2 \\ 3 & 1 & 3 \\ 2 & 1/3 & 1 \end{pmatrix}; \quad \mathbf{A}^{\mathrm{F}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \mathbf{A}^{\mathrm{S}} = \begin{pmatrix} 1 & 5 & 1 \\ 1/5 & 1 & 1/5 \\ 1 & 5 & 1 \end{pmatrix}$$

$$\mathbf{A}^{\mathrm{V}} = \begin{pmatrix} 1 & 9 & 7 \\ 1/9 & 1 & 1/5 \\ 1/7 & 5 & 1 \end{pmatrix}; \quad \mathbf{A}^{\mathrm{P}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix}; \quad \mathbf{A}^{\mathrm{M}} = \begin{pmatrix} 1 & 6 & 4 \\ 1/6 & 1 & 1/3 \\ 1/4 & 3 & 1 \end{pmatrix}$$

The collective multiplicative preference relation obtained using the linguistic quantifier "most of" with the corresponding weighting vector $\mathbf{W} = (0, 1/15, 1/3, 1/3, 4/15, 0)$ is

$$\mathbf{A}^{c} = \Phi_{\mathrm{most}}^{IC-G}(\mathbf{A}^{\mathrm{L}}, \mathbf{A}^{\mathrm{F}}, \mathbf{A}^{\mathrm{S}}, \mathbf{A}^{\mathrm{V}}, \mathbf{A}^{\mathrm{P}}, \mathbf{A}^{\mathrm{M}}) = \begin{pmatrix} 1 & 3.12 & 2.67 \\ 0.32 & 1 & 0.45 \\ 0.37 & 2.22 & 1 \end{pmatrix}$$

Remark 4. The collective multiplicative preference relation we have obtained in this example, following our tie procedure coincides with the one we would have obtained following Yager and Filev's procedure of replacing the arguments of the tied criteria $\mathbf{A}^{\mathrm{F}}$ and $\mathbf{A}^{\mathrm{M}}$ by their (geometric) average because the same weight is associated with $\mathbf{A}^{\mathrm{F}}$ and $\mathbf{A}^{\mathrm{M}}$ by the linguistic quantifier "most of" in the aggregation process. It is clear that when the weights associated with $\mathbf{A}^{\mathrm{F}}$ and $\mathbf{A}^{\mathrm{M}}$ are different, the procedure we propose gives different results to that of Yager and Filev's procedure.

5. RECIPROCITY AND CONSISTENCY PROPERTIES OF THE COLLECTIVE MULTIPLICATIVE PREFERENCE RELATION

In GDM models with ratio-scale preference assessments, it usually is assumed that the multiplicative preference relations to express the judgements are reciprocal. However, it is well known that reciprocity generally is not maintained after aggregation is performed in the selection process.[22] In Example 5, the collective multiplicative preference relation obtained, using an I-IOWG operator, was reciprocal, and the one obtained in Example 6, using the C-IOWG operator, also was reciprocal. However, the one obtained by applying the P-IOWG operator was not reciprocal.

In the following, we will show that IOWG operators acting as WGM operators maintain both the reciprocity and the consistency properties. On the other hand, the IOWG operators acting as OWG operators generally do not maintain these properties.

5.1. Reciprocity Property

If a group of experts $E = \{e_1, \ldots, e_m\}$ provides preferences about the alternatives $X = \{x_1, \ldots, x_n\}$ by means of multiplicative preference relations $\{A^1, \ldots, A^m\}$, which are reciprocal, $a_{ij}^k \cdot a_{ji}^k = 1$, $\forall i, j, k$, and if $\{u_1, \ldots, u_m\}$ is a set of order-inducing (importance or consistency) values associated with the set of experts, then the collective multiplicative preference relation $\mathbf{A}^c = (a_{ij}^c)$, obtained by using an IOWG operator Φ_Q^{I-G}, guided by a linguistic quantifier Q is also reciprocal.

In fact,

$$a_{ij}^c = \Phi_{\mathbf{W}}^G(\langle u_1, a_{ij}^1\rangle, \ldots, \langle u_n, a_{ij}^m\rangle) = \prod_{k=1}^{m} (a_{ij}^{\sigma(k)})^{w_k}$$

being $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ a permutation such that $u_{\sigma(k)} \geqslant u_{\sigma(k+1)}$, $\forall k = 1, \ldots, n - 1$.

It is clear that

$$a_{ji}^c = \Phi_{\mathbf{W}}^G(\langle u_1, a_{ji}^1\rangle, \ldots, \langle u_n, a_{ji}^m\rangle) = \prod_{k=1}^{m} (a_{ji}^{\sigma(k)})^{w_k} = \prod_{k=1}^{m} \left(\frac{1}{a_{ij}^{\sigma(k)}}\right)^{w_k} = \frac{1}{\prod_{k=1}^{m} (a_{ij}^{\sigma(k)})^{w_k}} = \frac{1}{a_{ij}^c}$$

i.e., $\mathbf{A}^c$ verifies reciprocity property.

5.2. Consistency Property

If the set of multiplicative preference relations are consistent,[27] i.e.,

$$a_{ij}^r \cdot a_{jk}^r = a_{ik}^r, \quad \forall i, j, k, r$$

in Ref. 22 we have shown that the ordering among the alternatives provided by Saaty's eigenvector method and the multiplicative choice degrees defined in Refs. 7, 20, and 21 are the same. Therefore, knowing how to maintain the consistency property in the aggregation process could be of great interest to a decision maker.

If $\mathbf{A}^c = \Phi_{\mathbf{W}}^G(\langle u_1, \mathbf{A}^1\rangle, \ldots, \langle u_n, \mathbf{A}^m\rangle)$, then

$$a_{ij}^c = \prod_{r=1}^{m} (a_{ij}^{\sigma(r)})^{w_r}; \quad a_{jk}^c = \prod_{r=1}^{m} (a_{jk}^{\sigma(r)})^{w_r}; \quad a_{ik}^c = \prod_{r=1}^{m} (a_{ik}^{\sigma(r)})^{w_r}$$

with $u_{\sigma(r)}$ the rth highest value in the set $\{u_1, \ldots, u_m\}$. Thus, we obtain

$$a_{ij}^c \cdot a_{jk}^c = \prod_{r=1}^{m} (a_{ij}^{\sigma(r)})^{w_r} \cdot \prod_{r=1}^{m} (a_{jk}^{\sigma(r)})^{w_r} = \prod_{r=1}^{m} (a_{ij}^{\sigma(r)} \cdot a_{jk}^{\sigma(r)})^{w_r} = \prod_{r=1}^{m} (a_{ik}^{\sigma(r)})^{w_r} = a_{ik}^c$$

which proves the consistency of $\mathbf{A}^c$.

Remark 5. The proof of the reciprocity and consistency of the collective multiplicative preference relation is based on the assumption that the order-inducing values are unchanged. If this is not the case, then the reciprocity and consistency do not necessarily hold. In fact, for the subclass of the OWG operators, we have shown in Ref. 22 that reciprocity is not maintained in the aggregation process and that there exists OWG operators that although maintain reciprocity do not maintain consistency.

6. CONCLUDING REMARKS

In this study we have introduced the IOWG operators and have studied their properties. We have shown that the WGM and OWG operators are subclasses of the IOWG. We have given examples of the use of IOWG operators in the aggregation of multiplicative preference relations. We have defined three IOWG operators that implement a semantic meaning in the aggregation process: the I-IOWG operator, which induces the ordering of the argument values based on the importance of the information sources; the C-IOWG operator, which induces the ordering of the argument values based on the consistency of the information sources; and the P-IOWG operator, which induces the ordering of the arguments based on the relative preference associated with each one of them. We also have given a sequential procedure to deal with ties in respect to the ordering induced by the application of one of these IOWG operators, different from the one proposed by Yager and Filev, consisting in a sequential application of the foregoing IOWG operators. The application of this sequential procedure induces an ordering of the arguments to be aggregated without ties or, in the extreme case of their presence, these do not affect the aggregated result. Finally, we have shown that the collective multiplicative preference relation verifies the reciprocity and consistency properties under the assumption that the order-inducing values remain unchanged.

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