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An optimization-based approach to adjusting unbalanced linguistic preference relations to obtain a required consistency level

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ABSTRACT

In certain real decision-making situations, decision makers often feel more comfortable providing their knowledge and preferences in linguistic terms. Unbalanced linguistic term sets may be used in decision problems with preference relations. However, the lack of consistency in decision-making with linguistic preference relations can lead to inconsistent conclusions. Based on the consistency measure of unbalanced linguistic preference relations, this paper proposes an optimization-based approach to improving the consistency level of unbalanced linguistic preference relations. This consistency-improving model preserves the utmost original knowledge and preferences in the process of improving consistency. Furthermore, it guarantees that the elements in the optimal adjusted unbalanced linguistic preference relation are all simple unbalanced linguistic terms. Finally, we propose a mixed 0–1 linear programming aimed to obtain the optimum solution to the proposed consistency improving model and to demonstrate its practicability.

1. Introduction

Preference relations are popular techniques used to model decision makers' knowledge and preferences regarding decision problems. There are two varieties of preference relations: numerical preference relations [7,14,20,22,31,33,37] and linguistic preference relations [10,24,35]. The consistency issue is a very important problem in decision-making with preference relations. Since the lack of consistency may lead to inconsistent conclusions [15,20,23,38].

In some real decision-making situations, decision makers often feel more comfortable providing their knowledge and preferences linguistically. Solving a decision problem with linguistic information implies the need for computing with words [16,24,25,28]. In particular, Herrera and Martínez [17] proposed the 2-tuple linguistic representation model to deal with uniformly and symmetrically distributed linguistic term sets. There has been much research on the question of consistency measures (e.g., [2,3,6,9,34]) regarding 2-tuple linguistic preference relations. Alonso et al. [2,3] and Cabrerizo et al. [6] proposed an interesting consistency index based on additive transitivity and linguistic 2-tuples. Dong et al. [9] developed a 2-tuple linguistic index to measure the consistency degree of linguistic preference relations. The Dong et al. [9] index not only measures the consistency degree of linguistic preference relations linguistically, but also reflects individual differences in the consistency degree of linguistic preference relations.

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However, the 2-tuple linguistic representation model only guarantees accuracy when dealing with a uniformly and symmetrically distributed linguistic term set. In recent years, the methodologies based on linguistic 2-tuples have been developed to deal with term sets that are non-uniformly and non-symmetrically distributed [1,4,11–13,19,21,26,32]. Herrera et al. [19] developed an interesting model called unbalanced linguistic term sets to handle such term sets. Nonetheless, there is little research on the question of consistency issues of unbalanced linguistic preference relations. Cabrerizo et al. [5] proposed the additive transitivity for unbalanced linguistic preference relations and presented a consistency index to measure the degree of consistency of unbalanced linguistic preference relations. However, Cabrerizo et al. [5] did not discuss how to improve the consistency level.

In this study, we discuss the consistency of unbalanced linguistic preference relations with the aim of obtaining a modified unbalanced linguistic preference relation with a required consistency level. To undertake this, we propose an optimization-based approach to obtaining an optimum solution. This optimization-based model preserves the utmost original preference information in the process of improving the consistency, according to the required consistency level. Moreover, it guarantees that the elements of the optimal adjusted unbalanced linguistic preference relations are all presented in simple unbalanced linguistic terms. Meanwhile, we demonstrate that a mixed 0–1 linear programming can be used to obtain the optimum solution to this optimization-based model.

The remainder of this paper is organized as follows: Section 2 introduces the 2-tuple linguistic methodology to deal with unbalanced linguistic term sets [19], as well as the consistency measure of unbalanced linguistic preference relations presented in Cabrerizo et al. [5]. This is followed by Section 3 that presents an optimization-based approach to improving the consistency level in unbalanced linguistic preference relations. Section 4 then provides an illustrative example. Section 5 then concludes this paper with final remarks.

2. Preliminaries

This section introduces the 2-tuple linguistic methodology dealing with unbalanced linguistic term sets [19] and presents the consistency measure of unbalanced linguistic preference relations in Cabrerizo et al. [5] that provides the basis for this study.

2.1. Linguistic methodology to deal with unbalanced linguistic term sets

The basic notations and operational laws of linguistic variables are introduced in [19,25,27,30,36]. Let $S = \{s_i | i = 0, 1, \ldots, g\}$ be a linguistic term set with odd cardinality. The term s_i represents a possible value for a linguistic variable and it is required that the linguistic term set is ordered: $s_i > s_j$ if and only if $i > j$.

The 2-tuple linguistic representation model, presented in Herrera and Martínez [17], represents the linguistic information by a 2-tuple (s_i, α), where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. This linguistic model defines a function with the purpose of making transformations between linguistic 2-tuples and numerical values. Formally, let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation. Then the 2-tuple that expresses the equivalent information to β is obtained by means of the following function:

$$\Delta : [0, g] \to S \times [-0.5, 0.5) \tag{1}$$

where

$$\Delta(\beta) = (s_i, \alpha), with \begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases}. \tag{2}$$

Clearly, Δ is a one to one mapping function. For convenience, its range is denoted as $\overline{S}$. Accordingly, Δ has an inverse function with $\Delta^{-1} : \overline{S} \to [0, g]$ and $\Delta^{-1}(s_i, \alpha) = i + \alpha$. In this study, (s_i, α) is denoted a simple term if $\alpha = 0$.

Unbalanced linguistic term sets have been proposed in [19]. Generally, an unbalanced linguistic term set S has a minimum term, a maximum term and a central term, and the remaining terms are non-uniformly and non-symmetrically distributed around the central one on both left and right lateral sets, i.e.,

$$S = S_L \cup S_C \cup S_R \tag{3}$$

where the left lateral set S_L contains all the terms smaller than the central term, where the central set S_C contains only the central term, and where the right lateral set S_R contains all the terms higher than the central term.

The concept of linguistic hierarchies [8,18], i.e., $LH = \cup_t l(t, n(t))$, is used to obtain 2-tuple linguistic representations of unbalanced linguistic values without a loss of information. A linguistic hierarchy is a set of levels in which each level is a linguistic term set with different granularity from the remaining levels of the hierarchy. Each level belonging to a linguistic hierarchy is denoted as $l(t, n(t))$, with t being a number that indicates the level of the hierarchy and s_5 the granularity of the linguistic term set of t. Generally, the linguistic term set $S^{n(t+1)}$ of level $t + 1$ is obtained from its predecessor $S^{n(t)}$ as $l(t, n(t)) \to l(t + 1, 2, \ldots, n(t) - 1)$. In linguistic hierarchies LH, the transformation function between terms from different levels to represent 2-tuple linguistic representations is as follows [18]: for any linguistic levels t and $t', TF^t_{t'} : l(t, n(t)) \to l(t', n(t'))$, so that

$$TF_{t'}^{t}\left(s_i^{n(t)}, \alpha^{n(t)}\right) = \Delta\left(\frac{\Delta^{-1}\left(s_i^{n(t)}, \alpha^{n(t)}\right) \cdot (n(t') - 1)}{n(t) - 1}\right). \tag{4}$$

In the computational model defined for the linguistic hierarchy LH, the unbalanced linguistic term set S, whose terms have different levels in LH, should be first transformed into the terms of the same level in LH. In this study, the maximum level t' in the LH is used, i.e., $l(t', n(t')) = S^{n(t')} = \left\{s_0^{n(t')}, s_1^{n(t')}, \ldots, s_{n(t')-1}^{n(t')}\right\}$.

For any 2-tuple linguistic representation (s_i, α), it can be transformed by the unbalanced linguistic transformation process into the term in $LH = \cup_t l(t, n(t))$ and vice versa. The detailed transformation process is described below:

(1) Representation in the linguistic hierarchy: the representation algorithm uses the linguistic hierarchy LH to model the unbalanced terms in S. Therefore, the first step towards accomplishing the process of computing with words is to transform the unbalanced terms in s_7 into their corresponding terms in the LH, by means of the transformation function ψ associating each unbalanced linguistic 2-tuple (s_i, α) with its respective linguistic 2-tuple in $LH(\overline{S})$, i.e.,

$$\psi : \overline{S} \to LH(\overline{S}), \tag{5}$$

so that $\psi(s_i, \alpha) = \left(s_{I(i)}^{G(i)}, \lambda\right)$, for $\forall (s_i, \alpha) \in \overline{S}$.

(2) Computational phase: this accomplishes the process of computing with words by using the computation model defined for the linguistic hierarchy. First it uses Eq. (4) to transform $\left(s_{I(i)}^{G(i)}, \lambda\right)$ into linguistic 2-tuples, denoted as $\left(s_{I'(i)}^{n(t')}, \lambda'\right)$ in $\overline{S}^{n(t')}$, i.e.,

$$\left(s_{I'(i)}^{n(t')}, \lambda'\right) = \Delta\left(\frac{\Delta^{-1}\left(s_{I(i)}^{G(i)}, \lambda\right) \cdot (n(t') - 1)}{G(i) - 1}\right). \tag{6}$$

Then, the computational model developed for the 2-tuple linguistic representation model is used over $\overline{S}^{n(t')}$ with a result denoted as $\left(s_r^{n(t')}, \lambda_r\right) \in \overline{S}^{n(t')}$.

(3) Retranslation process: A retranslation process is used to transform the result $\left(s_r^{n(t')}, \lambda_r\right) \in \overline{S}^{n(t')}$ into the unbalanced term in s_5, by using the transformation function ψ^{-1}, i.e.,

$$\psi^{-1} : LH(\overline{S}) \to \overline{S}, \tag{7}$$

so that $\psi^{-1}\left(s_r^{n(t')}, \lambda_r\right) = (s_{result}, \lambda_{result}) \in \overline{S}$.

The details of the methodology to deal with unbalanced linguistic term sets are described in Herrera et al. [19].

2.2. Consistency measure of unbalanced linguistic preference relations

In this subsection, we introduce the consistency measure of unbalanced linguistic preference relations, presented in Cabrerizo et al. [5].

Let $A = \{A_1, A_2, \ldots, A_m\}$ $(n \geqslant 2)$ be the set of alternatives to be evaluated. When a decision maker makes pairwise comparisons using the unbalanced linguistic term set S, he/she can construct an unbalanced linguistic preference relation $L = (l_{ij})_{n\times n}$, where l_{ij} denotes the unbalanced linguistic preference degree of the alternative b_2 over A_j. The unbalanced linguistic preference relation can be defined as Definition 1.

Definition 1 [5]. Let $S = \{s_0, s_1, \ldots, s_g\}$ be the unbalanced linguistic term set. The unbalanced linguistic preference relation L over a set of alternatives A is characterized by a membership function u_L:$A \times A \to S$, where $u_L(A_i, A_j) = l_{ij}$, $\forall i, j \in \{1, 2, \ldots, n\}$ and $l_{ij} \in S$.

In decision-making problems, it is not always possible for decision makers to provide all the possible preference assessments over the set of alternatives [2,3,5,20]. Moreover, considering the characteristics of an unbalanced linguistic term set, the unbalanced linguistic preference relation may not satisfy the reciprocity. Therefore, the lack of information in constructing unbalanced linguistic preference relations should be considered in this study. If some of the elements in an unbalanced linguistic preference relation cannot be provided by the decision maker, and denoted by *null*, then the preference relation is an incomplete unbalanced linguistic preference relation. For notational simplicity, in this study incomplete unbalanced linguistic preference relations will also be called unbalanced linguistic preference relations.

Definition 2 [5]. Let $L = (l_{ij})_{n\times n}$ be an unbalanced linguistic preference relation based on S, and let $TF_{t'}^{t}$ be the transformation function in a linguistic hierarchy. If for any l_{ij}, l_{jk}, $l_{ik} \neq null$,

$$l_{ik} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF_{t'}^{t}(LH(l_{ij}))\right) + \Delta^{-1}\left(TF_{t'}^{t}(LH(l_{jk}))\right) - \Delta^{-1}\left(TF_{t'}^{t}(LH(s_c))\right)\right)\right], \tag{8}$$

where s_c is the central term of S, then L is the unbalanced linguistic preference relation with the additive transitivity.

In Cabrerizo et al. [5], Eq. (8) is applied to calculate an estimated value of a preference degree using other preference degrees in an unbalanced linguistic preference relation. Specifically, the preference value $l_{ik}(i \neq k)$ is estimated using an intermediate alternative A_j in the following three different ways:

(1) From $l_{ik} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF^t_{t'}(LH(l_{ij}))\right) + \Delta^{-1}\left(TF^t_{t'}(LH(l_{jk}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(s_c))\right)\right)\right]$, the estimate of l_{ik}, denoted as el^{j1}_{ik}, is obtained. Namely, if l_{ij}, $l_{jk} \neq null$, then

$$el^{j1}_{ik} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF^t_{t'}(LH(l_{ij}))\right) + \Delta^{-1}\left(TF^t_{t'}(LH(l_{jk}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(s_c))\right)\right)\right]. \tag{9}$$

(2) From $l_{jk} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF^t_{t'}(LH(l_{ji}))\right) + \Delta^{-1}\left(TF^t_{t'}(LH(l_{ik}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(s_c))\right)\right)\right]$, the estimate of l_{ik}, denoted as el^{j2}_{ik}, is obtained. Namely, if l_{jk}, $l_{ji} \neq null$, then

$$el^{j2}_{ik} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF^t_{t'}(LH(l_{jk}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(l_{ji}))\right) + \Delta^{-1}\left(TF^t_{t'}(LH(s_c))\right)\right)\right]. \tag{10}$$

(3) From $l_{ij} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF^t_{t'}(LH(l_{ik}))\right) + \Delta^{-1}\left(TF^t_{t'}(LH(l_{kj}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(s_c))\right)\right)\right]$, the estimate of l_{ik}, denoted as el^{j3}_{ik}, is obtained. Namely, if l_{ij}, $l_{kj} \neq null$, then

$$el^{j3}_{ik} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF^t_{t'}(LH(l_{ij}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(l_{kj}))\right) + \Delta^{-1}\left(TF^t_{t'}(LH(s_c))\right)\right)\right]. \tag{11}$$

The consistency index of unbalanced linguistic preference relations, presented in Cabrerizo et al. [5], is as follows:

(1) Computing the consistency degree of a pair of alternatives.
For notational simplicity, let $M^1_{ik} = \{j | j \neq i, k; el^{j1}_{ik} \neq null\}, M^2_{ik} = \{j | j \neq i, k; el^{j2}_{ik} \neq null\}, M^3_{ik} = \{j | j \neq i, k; el^{j3}_{ik} \neq null\}$. Let $\#M^1_{ik}, \#M^2_{ik}$ and $\#M^3_{ik}$ denote, respectively, the number of the elements in the sets M^1_{ik}, M^2_{ik} and M^3_{ik}.
The error between a preference value l_{ik} and its estimated one el_{ik} is computed as

$$\varepsilon l_{ik} = \frac{\left|\Delta^{-1}\left(TF^t_{t'}(LH(el_{ik}))\right) - \Delta^{-1}\left(TF^t_{t'}(LH(l_{ik}))\right)\right|}{n(t') - 1}, \tag{12}$$

where,

$$el_{ik} = LH^{-1}\left[\Delta\left(\frac{\sum_{j \in M^1_{ik}} \Delta^{-1}\left(TF^t_{t'}\left(LH\left(el^{j1}_{ik}\right)\right)\right) + \sum_{j \in M^2_{ik}} \Delta^{-1}\left(TF^t_{t'}(LH(el^{j2}_{ik}))\right) + \sum_{j \in M^3_{ik}} \Delta^{-1}\left(TF^t_{t'}\left(LH\left(el^{j3}_{ik}\right)\right)\right)}{\#M^1_{ik} + \#M^2_{ik} + \#M^3_{ik}}\right)\right]. \tag{13}$$

Then the consistency level associated to a preference value l_{ik} is defined as

$$CL_{ik} = 1 - \varepsilon l_{ik}. \tag{14}$$

(2) Computing the consistency degree of a relation.
The consistency level of an unbalanced linguistic preference relation L is defined as

$$CL(L) = \frac{\sum_{C_{ik} \in H} CL_{ik}}{\#H}, \tag{15}$$

where $H = \{CL_{ik} | CL_{ik} \neq null\}$. When $CL(L) = 1$, the unbalanced linguistic preference relation L is fully consistent. Otherwise, the lower the $CL(L)$, the more inconsistent the L.

3. Optimization-based consistency improving model

In this section, we propose an optimization-based model with the aim of obtaining a modified unbalanced linguistic preference relation with a required consistency level. In addition, we show how a mixed 0–1 linear programming can be used to obtain the optimum solution to this optimization-based model.

3.1. Optimization-based model to obtain an established consistency level

Let $S = \{s_0, s_1, \ldots, s_g\}$ be an unbalanced linguistic term set. Let L be an unbalanced linguistic preference relation based on S. The principal aim of dealing with inconsistency in $L = (l_{ij})_{n \times n}$ is to find a suitable unbalanced linguistic preference relation $\overline{L} = (\overline{l_{ij}})_{n \times n}$ with the established consistency index $\overline{CL}$.

This study uses the Manhattan distance to measure the distance between two unbalanced linguistic preference relations, i.e., the distance between L and $\overline{L}$ is defined, as in Cabrerizo et al. [5],

$$d(L,\overline{L}) = \sum_{l_{ij}\neq null} \frac{\left|\Delta^{-1}(TF^t_{t'}(LH(l_{ij}))) - \Delta^{-1}\left(TF^t_{t'}(LH(\overline{l_{ij}}))\right)\right|}{n(t')-1}. \tag{16}$$

In order to preserve the most information possible in L, we hope that the distance measure between L and $\overline{L}$ is minimal, namely, $\min d(L,\overline{L})$, i.e.,

$$\min_{\overline{L}} \sum_{l_{ij}\neq null} \frac{\left|\Delta^{-1}(TF^t_{t'}(LH(l_{ij}))) - \Delta^{-1}\left(TF^t_{t'}(LH(\overline{l_{ij}}))\right)\right|}{n(t')-1}. \tag{17}$$

At the same time, $\overline{L}$ must have the established consistency index, namely,

$$CL(\overline{L}) \geqslant \overline{CL}. \tag{18}$$

In order to accord with the expression of the decision makers, it is our hope that the elements of the optimal adjusted unbalanced linguistic preference relation are all simple unbalanced linguistic terms, i.e., $\overline{l_{ij}} \in S$.

Based on (17) and (18), the optimization model to improve the consistency of L can be constructed as follows:

$$\begin{cases} \min d(L,\overline{L}) \\ s.t.\ CL(\overline{L}) \geqslant \overline{CL} \\ \overline{l_{ij}} \in S \end{cases} \tag{19}$$

We denote model (19) as P_1. We noted that in this study $\overline{l_{ij}} = null$ if $l_{ij} = null$.

3.2. Mixed 0–1 linear programming associated with the optimization-based model

In this subsection, we construct a mixed 0–1 linear programming to obtain the optimum solution to the consistency improving model P_1.

First, for any $\overline{l_{ij}} \neq null$, let

$$x^r_{ij} = \begin{cases} 0 & if\ \overline{l_{ij}} \neq s_r \\ 1 & if\ \overline{l_{ij}} = s_r \end{cases}, \quad (i,j = 1,2\ldots,n; r = 0,1,\ldots,g). \tag{20}$$

Clearly, $x^r_{ij} \in \{0,1\}$ and $\sum_{r=0}^{g} x^r_{ij} = 1$ if $\overline{l_{ij}} \neq null$. In this manner, $\overline{l_{ij}}$ can be equivalently expressed by x^r_{ij}. For example, if $\left\{x^0_{ij}, x^1_{ij}, \ldots, x^g_{ij}\right\} = \{0,1,0,\ldots,0\}$, then $\overline{l_{ij}} = s_1$. Otherwise, if $\overline{l_{ij}} = s_3$, then $\left\{x^0_{ij}, x^1_{ij}, x^2_{ij}, x^3_{ij}, \ldots, x^g_{ij}\right\} = \{0,0,0,1,0,\ldots,0\}$.

Let

$$s'_r = TF^t_{t'}(LH(s_r))\ (r = 0,1,\ldots,g), \tag{21}$$
$$l'_{ij} = TF^t_{t'}(LH(l_{ij}))\ (i,j = 0,1,\ldots,n), \tag{22}$$

and

$$\overline{l_{ij}}' = TF^t_{t'}(LH(\overline{l_{ij}}))\ (i,j = 0,1,\ldots,n). \tag{23}$$

Lemma 1. *For any* $\overline{l_{ij}} \neq null$, $\Delta^{-1}\left(\overline{l_{ij}}'\right) = \sum_{r=0}^{g}\left(x^r_{ij}\cdot\Delta^{-1}(s'_r)\right)$.

Proof. Without loss of generality, we assume that $\overline{l_{ij}} = s_k \in S$. Then, according to Eq. (20), $x^k_{ij} = 1$ and $\sum_{r=0,r\neq k}^{g} x^r_{ij} = 0$. Hence, we obtain

$$\Delta^{-1}\left(\overline{l_{ij}}'\right) = \Delta^{-1}(s'_k) = x^k_{ij}\cdot\Delta^{-1}(s'_k) + \sum_{r=0,r\neq k}^{g}\left(x^r_{ij}\cdot\Delta^{-1}(s'_r)\right) = \sum_{r=0}^{g}\left(x^r_{ij}\cdot\Delta^{-1}(s'_r)\right). \tag{24}$$

As a result, $\Delta^{-1}\left(\overline{l_{ij}}'\right) = \Delta^{-1}(s'_k) = \sum_{r=0}^{g}\left(x^r_{ij}\cdot\Delta^{-1}(s'_r)\right)$. This completes the proof of Lemma1. □

Let $\overline{el^{j1}_{ik}}, \overline{el^{j2}_{ik}}$ and $\overline{el^{j3}_{ik}}$ be the estimated values of $\overline{l_{ik}}$ obtained by additive transitivity, according to Eqs. (9)–(11). Let

$$\overline{el^{j1}_{ik}}' = TF^t_{t'}\left(LH\left(\overline{el^{j1}_{ik}}\right)\right), \tag{25}$$

$$\overline{el^{j2}_{ik}}' = TF^t_{t'}\left(LH\left(\overline{el^{j2}_{ik}}\right)\right), \tag{26}$$

and

$$\overline{el_{ik}^{j3}}' = TF_{t'}^{t}\left(LH\left(\overline{el_{ik}^{j3}}\right)\right). \tag{27}$$

Lemma 2.

(i) *If* $\overline{l_{ij}}, \overline{l_{jk}} \neq null$, *then* $\Delta^{-1}\left(\overline{el_{ik}^{j1}}'\right) = \sum_{r=0}^{g}\left(\left(x_{ij}^{r} + x_{jk}^{r}\right) \cdot \Delta^{-1}(s_r')\right) - \Delta^{-1}(s_c')$.

(ii) *If* $\overline{l_{jk}}, \overline{l_{ji}} \neq null$, *then* $\Delta^{-1}\left(\overline{el_{ik}^{j2}}'\right) = \sum_{r=0}^{g}\left(\left(x_{jk}^{r} - x_{ji}^{r}\right) \cdot \Delta^{-1}(s_r')\right) + \Delta^{-1}(s_c')$.

(iii) *If* $\overline{l_{ij}}, \overline{l_{kj}} \neq null$, *then* $\Delta^{-1}\left(\overline{el_{ik}^{j3}}'\right) = \sum_{r=0}^{g}\left(\left(x_{ij}^{r} - x_{kj}^{r}\right) \cdot \Delta^{-1}(s_r')\right)) + \Delta^{-1}(s_c')$.

Proof. Here, we only prove (i), and the proofs for (ii) and (iii) are similar. According to Eq. (9), if $\overline{l_{ij}}, \overline{l_{jk}} \neq null$, we can obtain,

$$\overline{el_{ik}^{j1}} = LH^{-1}\left[\Delta\left(\Delta^{-1}\left(TF_{t'}^{t}(LH(\overline{l_{ij}}))\right) + \Delta^{-1}\left(TF_{t'}^{t}(LH(\overline{l_{jk}}))\right) - \Delta^{-1}(TF_{t'}^{t}(LH(s_c)))\right)\right].$$

Using Eqs. (21), (23) and (25) we arrive at $\Delta^{-1}\left(\overline{el_{ik}^{j1}}'\right) = \Delta^{-1}\left(\overline{l_{ij}}'\right) + \Delta^{-1}\left(\overline{l_{jk}}'\right) - \Delta^{-1}(s_c')$. From Eq. (24), we arrive at $\Delta^{-1}\left(\overline{l_{ij}}'\right) = \sum_{r=0}^{g}\left(x_{ij}^{r} \cdot \Delta^{-1}(s_r')\right)$ and $\Delta^{-1}\left(\overline{l_{jk}}'\right) = \sum_{r=0}^{g}\left(x_{jk}^{r} \cdot \Delta^{-1}(s_r')\right)$. Therefore, $\Delta^{-1}\left(\overline{el_{ik}^{j1}}'\right) = \sum_{r=0}^{g}\left(x_{ij}^{r} \cdot \Delta^{-1}(s_r')\right) + \sum_{r=0}^{g}\left(x_{jk}^{r} \cdot \Delta^{-1}(s_r')\right) - \Delta^{-1}(s_c') = \sum_{r=0}^{g}\left(\left(x_{ij}^{r} + x_{jk}^{r}\right) \cdot \Delta^{-1}(s_r')\right) - \Delta^{-1}(s_c')$. This completes the proof of Lemma 2. □

Let $\overline{M}_{ik}^{1} = \left\{j \middle| j \neq i,k; \overline{el_{ik}^{j1}} \neq null\right\}, \overline{M}_{ik}^{2} = \{j \mid j \neq i,k; \overline{el_{ik}^{j2}} \neq null\}$ and $\overline{M}_{ik}^{3} = \left\{j \middle| j \neq i,k; \overline{el_{ik}^{j3}} \neq null\right\}$. Let $\#\overline{M}_{ik}^{1}, \#\overline{M}_{ik}^{2}$ and $\#\overline{M}_{ik}^{3}$ respectively denote the number of the elements in the sets $\overline{M}_{ik}^{1}, \overline{M}_{ik}^{2}$ and $\overline{M}_{ik}^{3}$.

Lemma 3. *Let* $\overline{el_{ik}^{j1}}', \overline{el_{ik}^{j2}}', \overline{el_{ik}^{j3}}'$ *be defined as before. Then,* $CL(\overline{L}) = 1 - \frac{\sum_{\varepsilon\overline{l_{ik}} \neq null} \varepsilon\overline{l_{ik}}}{\#V}$, *where* $V = \{\varepsilon\overline{l_{ik}} \mid \varepsilon\overline{l_{ik}} \neq null\}$ *and*

$$\varepsilon\overline{l_{ik}} = \frac{1}{n(t')-1}\left|\frac{\sum_{j\in\overline{M}_{ik}^{1}}\Delta^{-1}\left(\overline{el_{ik}^{j1}}'\right) + \sum_{j\in\overline{M}_{ik}^{2}}\Delta^{-1}\left(\overline{el_{ik}^{j2}}'\right) + \sum_{j\in\overline{M}_{ik}^{3}}\Delta^{-1}\left(\overline{el_{ik}^{j3}}'\right)}{\#\overline{M}_{ik}^{1} + \#\overline{M}_{ik}^{2} + \#\overline{M}_{ik}^{3}} - \sum_{r=0}^{g}(x_{ik}^{r} \cdot \Delta^{-1}(s_r'))\right|.$$

Proof. According to Eqs. (14) and (15), we can obtain $CL(\overline{L}) = \frac{\sum_{C\overline{L_{ik}} \neq null} C\overline{L_{ik}}}{\#V}$, where $C\overline{L_{ik}} = 1 - \varepsilon\overline{l_{ik}}$. So, $CL(\overline{L}) = 1 - \frac{\sum_{\varepsilon\overline{l_{ik}} \neq null} \varepsilon\overline{l_{ik}}}{\#V}$. Based on Eqs. (12) and (13), we obtain

$$\varepsilon\overline{l_{ik}} = \frac{1}{n(t')-1} \times \left|\frac{\sum_{j\in\overline{M}_{ik}^{1}}\Delta^{-1}\left(TF_{t'}^{t}\left(LH\left(\overline{el_{ik}^{j1}}\right)\right)\right) + \sum_{j\in\overline{M}_{ik}^{2}}\Delta^{-1}\left(TF_{t'}^{t}\left(LH\left(\overline{el_{ik}^{j2}}\right)\right)\right) + \sum_{j\in\overline{M}_{ik}^{3}}\Delta^{-1}\left(TF_{t'}^{t}\left(LH\left(\overline{el_{ik}^{j3}}\right)\right)\right)}{\#\overline{M}_{ik}^{1} + \#\overline{M}_{ik}^{2} + \#\overline{M}_{ik}^{3}}) - \Delta^{-1}\left(TF_{t'}^{t}(LH(\overline{l_{ik}}))\right)\right|.$$

Using Eqs. (24)–(27), the following can be obtained:

$$\varepsilon\overline{l_{ik}} = \frac{1}{n(t')-1}\left|\frac{\sum_{j\in\overline{M}_{ik}^{1}}\Delta^{-1}\left(\overline{el_{ik}^{j1}}'\right) + \sum_{j\in\overline{M}_{ik}^{2}}\Delta^{-1}\left(\overline{el_{ik}^{j2}}'\right) + \sum_{j\in\overline{M}_{ik}^{3}}\Delta^{-1}\left(\overline{el_{ik}^{j3}}'\right)}{\#\overline{M}_{ik}^{1} + \#\overline{M}_{ik}^{2} + \#\overline{M}_{ik}^{3}} - \sum_{r=0}^{g}(x_{ik}^{r} \cdot \Delta^{-1}(s_r'))\right|.$$

This completes the proof of Lemma 3. □

Then, based on Lemmas 1–3, we propose Theorem 1.

Theorem 1. *By introducing the 0–1 variable* $x_{ij}^{r} = \begin{cases} 0 & if\ \overline{l_{ij}} \neq s_r \\ 1 & if\ \overline{l_{ij}} = s_r \end{cases}$, *model* P_1 *can be equivalently transformed into a model* (28)–(35), *that is denoted* P_2.

$$\text{Min } \frac{1}{n(t')-1}\sum_{l_{ij}\neq null}\left|\Delta^{-1}\left(l_{ij}'\right) - \sum_{r=0}^{g}\left(x_{ij}^{r} \cdot \Delta^{-1}(s_r')\right)\right| \tag{28}$$

s.t.

$$x_{ij}^r \in \{0,1\}, i,j=1,2,\ldots,n; \quad l_{ij} \neq null; r=0,1,\ldots,g \tag{29}$$

$$\sum_{r=0}^{g} x_{ij}^r = 1, i,j=1,2,\ldots,n; \quad l_{ij} \neq null \tag{30}$$

$$\Delta^{-1}(e\overline{l_{ik}^{j1}}') = \sum_{r=0}^{g}\left(\left(x_{ij}^r + x_{jk}^r\right)\cdot \Delta^{-1}(s_r')\right) - \Delta^{-1}(s_c'), i,j,k=1,2,\ldots,n; \quad l_{ik}, l_{ij}, l_{jk} \neq null \tag{31}$$

$$\Delta^{-1}(e\overline{l_{ik}^{j2}}') = \sum_{r=0}^{g}\left(\left(x_{jk}^r - x_{ji}^r\right)\cdot \Delta^{-1}(s_r')\right) + \Delta^{-1}(s_c'), i,j,k=1,2,\ldots,n; \quad l_{ik}, l_{jk}, l_{ji} \neq null \tag{32}$$

$$\Delta^{-1}\left(e\overline{l_{ik}^{j3}}'\right) = \sum_{r=0}^{g}\left((x_{ij}^r - x_{kj}^r)\cdot \Delta^{-1}(s_r')\right) + \Delta^{-1}(s_c'), i,j,k=1,2,\ldots,n; \quad l_{ik}, l_{ij}, l_{kj} \neq null \tag{33}$$

$$\varepsilon\overline{l_{ik}} = \frac{1}{n(t')-1}\left|\frac{\sum_{j\in \overline{M}_{ik}^1}\Delta^{-1}\left(e\overline{l_{ik}^{j1}}'\right) + \sum_{j\in \overline{M}_{ik}^2}\Delta^{-1}\left(e\overline{l_{ik}^{j2}}'\right) + \sum_{j\in \overline{M}_{ik}^3}\Delta^{-1}\left(e\overline{l_{ik}^{j3}}'\right)}{\#\overline{M}_{ik}^1 + \#\overline{M}_{ik}^2 + \#\overline{M}_{ik}^3} - \sum_{r=0}^{g}(x_{ik}^r\cdot \Delta^{-1}(s_r'))\right|, i,k=1,2,\ldots,n; \quad l_{ik} \neq null \tag{34}$$

$$\sum_{\varepsilon\overline{l_{ik}}\in V} \varepsilon\overline{l_{ik}} \leqslant \#V\cdot(1-\overline{CL}), i,k=1,2,\ldots,n \tag{35}$$

Proof. Based on Eqs. (22)–(24), we obtain

$$\min\ d(L,\overline{L}) = \min_{\overline{L}} \sum_{l_{ij}\neq null} \frac{\left|\Delta^{-1}(TF_{t'}^{t}(LH(l_{ij}))) - \Delta^{-1}\left(TF_{t'}^{t}(LH(\overline{l_{ij}}))\right)\right|}{n(t')-1} = \min \frac{1}{n(t')-1}\sum_{l_{ij}\neq null}\left|\Delta^{-1}\left(l_{ij}'\right) - \sum_{r=0}^{g}\left(x_{ij}^r\cdot \Delta^{-1}(s_r')\right)\right|.$$

According to Lemmas 1–3, the constraint $CL(\overline{L}) \geqslant \overline{CL}$ in model P_1 can be equivalently expressed by constraints (31)–(35). So, model P_1 can be equivalently transformed into model P_2. This completes the proof of Theorem 1. □

Theorem 2. *By introducing the four transformed decision variables* $z_{ik} = \frac{1}{n(t')-1}\left[\frac{\sum_{j\in \overline{M}_{ik}^1}\Delta^{-1}(e\overline{l_{ik}^{j1}}') + \sum_{j\in \overline{M}_{ik}^2}\Delta^{-1}\left(e\overline{l_{ik}^{j2}}'\right) + \sum_{j\in \overline{M}_{ik}^3}\Delta^{-1}\left(e\overline{l_{ik}^{j3}}'\right)}{\#\overline{M}_{ik}^1 + \#\overline{M}_{ik}^2 + \#\overline{M}_{ik}^3} - \sum_{r=0}^{g}(x_{ik}^r\cdot \Delta^{-1}(s_r'))\right]$, $\varepsilon\overline{l_{ik}} = |z_{ik}|, e_{ij} = \Delta^{-1}\left(l_{ij}'\right) - \sum_{r=0}^{g} x_{ij}^r\cdot \Delta^{-1}(s_r')$ *and* $f_{ij} = |e_{ij}|$, *model* P_2 *can be equivalently transformed into the following mixed 0–1 linear programming model, denoted as* P_3.

$$\text{Min}\ \frac{1}{n(t')-1}\sum_{l_{ij}\neq null} f_{ij} \tag{36}$$

s.t.

$$x_{ij}^r \in \{0,1\}, i,j=1,2,\ldots,n; \quad l_{ij} \neq null; r=0,1,\ldots,g \tag{37}$$

$$\sum_{r=0}^{g} x_{ij}^r = 1, i,j=1,2,\ldots,n; \quad l_{ij} \neq null \tag{38}$$

$$e_{ij} = \Delta^{-1}\left(l_{ij}'\right) - \sum_{r=0}^{g}\left(x_{ij}^r\cdot \Delta^{-1}(s_r')\right), i,j=1,2,\ldots,n; \quad l_{ij} \neq null \tag{39}$$

$$e_{ij} \leqslant f_{ij}, i,j=1,2,\ldots,n; \quad l_{ij} \neq null \tag{40}$$

$$-e_{ij} \leqslant f_{ij}, i,j=1,2,\ldots,n; \quad l_{ij} \neq null \tag{41}$$

$$\Delta^{-1}\left(e\overline{l_{ik}^{j1}}'\right) = \sum_{r=0}^{g}\left(\left(x_{ij}^r + x_{jk}^r\right)\cdot \Delta^{-1}(s_r')\right) - \Delta^{-1}(s_c'), i,j,k=1,2,\ldots,n; \quad l_{ik}, l_{ij}, l_{jk} \neq null \tag{42}$$

$$\Delta^{-1}\left(e\overline{l_{ik}^{j2}}'\right) = \sum_{r=0}^{g}\left(\left(x_{jk}^r - x_{ji}^r\right)\cdot \Delta^{-1}(s_r')\right) + \Delta^{-1}(s_c'), i,j,k=1,2,\ldots,n; \quad l_{ik}, l_{jk}, l_{ji} \neq null \tag{43}$$

$$\Delta^{-1}\left(e\overline{l_{ik}^{j3}}'\right) = \sum_{r=0}^{g}\left(\left(x_{ij}^r - x_{kj}^r\right)\cdot \Delta^{-1}(s_r')\right) + \Delta^{-1}(s_c'), i,j,k=1,2,\ldots,n; \quad l_{ik}, l_{ij}, l_{kj} \neq null \tag{44}$$

$$z_{ik} = \frac{1}{n(t')-1}\left[\frac{\sum_{j\in \overline{M}_{ik}^1}\Delta^{-1}\left(e\overline{l_{ik}^{j1}}'\right) + \sum_{j\in \overline{M}_{ik}^2}\Delta^{-1}\left(e\overline{l_{ik}^{j2}}'\right) + \sum_{j\in \overline{M}_{ik}^3}\Delta^{-1}\left(e\overline{l_{ik}^{j3}}'\right)}{\#\overline{M}_{ik}^1 + \#\overline{M}_{ik}^2 + \#\overline{M}_{ik}^3} - \sum_{r=0}^{g}(x_{ik}^r\cdot \Delta^{-1}(s_r'))\right], i,k=1,2,\ldots,n; \quad l_{ik} \neq null \tag{45}$$

$$z_{ik} \leqslant \varepsilon\overline{l_{ik}}, i,k=1,2,\ldots,n; \quad l_{ik} \neq null \tag{46}$$

$$-z_{ik} \leqslant \varepsilon\overline{l_{ik}}, i,k=1,2,\ldots,n; \quad l_{ik} \neq null \tag{47}$$

$$\sum_{\varepsilon\overline{l_{ik}}\in V} \varepsilon\overline{l_{ik}} \leqslant \#V\cdot(1-\overline{CL}), i,k=1,2,\ldots,n \tag{48}$$

Proof. In model P_3, Constraints (39)–(41) enforce that $f_{ij} \geqslant |e_{ij}| = \left|\Delta^{-1}\left(l'_{ij}\right) - \sum_{r=0}^{g}\left(x_{ij}^r \cdot \Delta^{-1}(s'_r)\right)\right|$. According to the objective function (i.e., Eq. (36)), any feasible solution with $f_{ij} > |e_{ij}|$ is not the optimal solution to model P_3. Thus, constraints (39)–(41) guarantee that $f_{ij} = |e_{ij}|$. Constraints (42)–(48) guarantee that $CL(\bar{L}) \geqslant \overline{CL}$. Therefore, by introducing the transformed decision variables, model P_2 can be equivalently transformed into model P_3, completing the proof of Theorem 2. □

Observation 1. The proposed consistency improving model provides the two following advantages: (i) it preserves the utmost of the information in the original unbalanced linguistic preference relations, and (ii) it guarantees that the elements in the optimal adjusted unbalanced linguistic preference relation are all simple unbalanced linguistic terms.

Observation 2. According to Miller [29], an individual cannot simultaneously compare more than 7 ± 2 objects without producing confusion. So, the number of alternatives and the granularity of unbalanced linguistic term sets must belong to the interval [5,9]. As a result, the proposed mixed 0–1 linear programming is not a large-scale optimization problem. Generally, mixed 0–1 linear programming models with a few hundred binary variables can be effectively and rapidly solved by several software packages (e.g., Lingo).

4. Illustrative example

The example presented in Cabrerizo et al. [5] is used in this section to demonstrate our proposal. The unbalanced linguistic term set used in this example is as follows:

$$S = \{s_0 = none(N), s_1 = very\ low(VL), s_2 = low(L), s_3 = medium(M), s_4 = high(H),$$
$$s_5 = quite\ high(QH), s_6 = very\ high(VH), s_7 = total(T)\}.$$

Cabrerizo et al. [5] use the algorithm presented in Herrera et al. [19] to obtain the 2-tuple linguistic representation of each term of the unbalanced linguistic term set $S = S_L \cup S_C \cup S_R$, in a linguistic hierarchy. After the transformation process in Herrera et al. [19], the final 2-tuple linguistic representations of S_C in a linguistic hierarchy are obtained, i.e.,

$$S_L = \left\{s_0 \leftarrow s_0^9, s_1 \leftarrow s_1^9, s_2 \leftarrow \bar{s}_2^9 \cup \underline{s}_1^5\right\}, S_C = \left\{s_3 \leftarrow \bar{s}_2^5 \cup \underline{s}_4^9\right\},$$

$$S_R = \left\{s_4 \leftarrow s_5^9, s_5 \leftarrow s_6^9, s_6 \leftarrow s_7^9, s_7 \leftarrow s_8^9\right\}.$$

The unbalanced linguistic preference relation may not satisfy the reciprocity, therefore, without loss of generality, we take the upper triangular of one of the unbalanced linguistic preference relations presented in Cabrerizo et al. [5] and denote it as L,

$$L = \begin{pmatrix} - & VH & QH & M \\ - & - & M & VH \\ - & - & - & VL \\ - & - & - & - \end{pmatrix}.$$

In the following, we apply the methods employed in Sections 2 and 3 to compute and improve the consistency level of L.

4.1. Computing the consistency index of L

First, based on Eqs. (9)–(11), we recognize that.
$M_{12}^1 = null$, $M_{12}^2 = null$, $M_{12}^3 = \{3,4\}$, $M_{13}^1 = \{2\}$, $M_{13}^2 = null$, $M_{13}^3 = \{4\}$, $M_{14}^1 = \{2,3\}$, $M_{14}^2 = null, M_{14}^3 = null$, $M_{23}^1 = null$, $M_{23}^2 = \{1\}$, $M_{23}^3 = \{4\}$, $M_{24}^1 = \{3\}$, $M_{24}^2 = \{1\}$, $M_{24}^3 = null$, $M_{34}^1 = null$, $M_{34}^3 = null$ and $M_{34}^2 = \{1,2\}$.
Obviously, $\#M_{ik}^1 + \#M_{ik}^2 + \#M_{ik}^3 = 2 \quad (i = 1,2,3, i+1 \leqslant k \leqslant 4)$.
In this study, we set the maximum level at $t' = 3$ and $n(t') = 9$. Let $s'_r = TF_{t'}^t(LH(s_r))$. Then, we obtain

$$s'_0 = s_0^9,\quad s'_1 = s_1^9,\quad s'_2 = s_2^9,\quad s'_3 = s_4^9,\quad s'_4 = s_5^9,\quad s'_5 = s_6^9,\quad s'_6 = s_7^9, \text{ and } s'_7 = s_8^9.$$

Next, utilizing Eqs. (9)–(11) and (13) to calculate the estimated values el_{ik} of l_{ik} $(i = 1, 2, 3, i + 1 \leqslant k \leqslant 4)$, we obtain
$el_{12} = (M, -0.25)$, $el_{13} = (VH, 0)$, $el_{14} = (VH, -0.5)$, $el_{23} = (VH, -0.5)$, $el_{24} = (VL, 0)$ and $el_{34} = (H, -0.5)$.
According to Eqs. (12) and (14), the error εl_{ik} between l_{ik} and its estimated value el_{ik} is computed as.
$\varepsilon l_{12} = 0.4375$, $\varepsilon l_{13} = 0.125$, $\varepsilon l_{14} = 0.3125$, $\varepsilon l_{23} = 0.3125$, $\varepsilon l_{24} = 0.75$ and $\varepsilon l_{34} = 0.4375$.
It is obvious that

$$CL_{12} = 0.5625, CL_{13} = 0.875,\quad CL_{14} = 0.6875,\quad CL_{23} = 0.6875,\quad CL_{24} = 0.25 \text{ and } CL_{34} = 0.5625.$$

Finally, we can obtain the consistency level of L by utilizing Eq. (15), i.e.,

$$CL(L) = \frac{\sum_{C_{ik} \in H} CL_{ik}}{\#H} = \frac{3.625}{6} = 0.604.$$

4.2. Improving the consistency index of L

As stated above, if $l_{ik} \neq null$, then $\overline{l_{ik}} \neq null$. Thus, in this example we have $\overline{M}_{ik}^1 = M_{ik}^1, \overline{M}_{ik}^2 = M_{ik}^2, \overline{M}_{ik}^3 = M_{ik}^3$ and $\#\overline{M}_{ik}^1 + \#\overline{M}_{ik}^2 + \#\overline{M}_{ik}^3 = 2 (i = 1,2,3, i+1 \leqslant k \leqslant 4)$.

We determine that $\overline{CL} = 0.7$. Since $CL(L) = 0.604 < 0.7$, a mixed 0–1 linear programming model (49)–(70) is used to improve the consistency of L.

Min

$$\begin{aligned}\frac{1}{8}\Big[&|7-(x_{12}^1+2x_{12}^2+4x_{12}^3+5x_{12}^4+6x_{12}^5+7x_{12}^6+8x_{12}^7)|+|6-(x_{13}^1+2x_{13}^2+4x_{13}^3+5x_{13}^4+6x_{13}^5+7x_{13}^6+8x_{13}^7)|\\&+|4-(x_{14}^1+2x_{14}^2+4x_{14}^3+5x_{14}^4+6x_{14}^5+7x_{14}^6+8x_{14}^7)|+|4-(x_{23}^1+2x_{23}^2+4x_{23}^3+5x_{23}^4+6x_{23}^5+7x_{23}^6+8x_{23}^7)|\\&+|7-(x_{24}^1+2x_{24}^2+4x_{24}^3+5x_{24}^4+6x_{24}^5+7x_{24}^6+8x_{24}^7)|+|1-(x_{34}^1+2x_{34}^2+4x_{34}^3+5x_{34}^4+6x_{34}^5+7x_{34}^6+8x_{34}^7)|\Big]\end{aligned} \tag{49}$$

s.t.

$$x_{ij}^0+x_{ij}^1+x_{ij}^2+x_{ij}^3+x_{ij}^4+x_{ij}^5+x_{ij}^6+x_{ij}^7=1,\quad i=1,2,3, i+1\leqslant j\leqslant 4 \tag{50}$$

$$x_{ij}^r \in \{0,1\},\quad i=1,2,3, i+1\leqslant j\leqslant 4, r=0,1,\ldots,7 \tag{51}$$

$$\Delta^{-1}\left(el_{12}^{\overline{33}\prime}\right)=x_{13}^1-x_{23}^1+2x_{13}^2-2x_{23}^2+4x_{13}^3-4x_{23}^3+5x_{13}^4-5x_{23}^4+6x_{13}^5-6x_{23}^5+7x_{13}^6-7x_{23}^6+8x_{13}^7-8x_{23}^7+4 \tag{52}$$

$$\Delta^{-1}\left(el_{12}^{\overline{43}\prime}\right)=x_{14}^1-x_{24}^1+2x_{14}^2-2x_{24}^2+4x_{14}^3-4x_{24}^3+5x_{14}^4-5x_{24}^4+6x_{14}^5-6x_{24}^5+7x_{14}^6-7x_{24}^6+8x_{14}^7-8x_{24}^7+4 \tag{53}$$

$$\Delta^{-1}\left(el_{13}^{\overline{21}\prime}\right)=x_{12}^1+x_{23}^1+2x_{12}^2+2x_{23}^2+4x_{12}^3+4x_{23}^3+5x_{12}^4+5x_{23}^4+6x_{12}^5+6x_{23}^5+7x_{12}^6+7x_{23}^6+8x_{12}^7+8x_{23}^7-4 \tag{54}$$

$$\Delta^{-1}\left(el_{13}^{\overline{43}\prime}\right)=x_{14}^1-x_{34}^1+2x_{14}^2-2x_{34}^2+4x_{14}^3-4x_{34}^3+5x_{14}^4-5x_{34}^4+6x_{14}^5-6x_{34}^5+7x_{14}^6-7x_{34}^6+8x_{14}^7-8x_{34}^7+4 \tag{55}$$

$$\Delta^{-1}\left(el_{14}^{\overline{21}\prime}\right)=x_{12}^1+x_{24}^1+2x_{12}^2+2x_{24}^2+4x_{12}^3+4x_{24}^3+5x_{12}^4+5x_{24}^4+6x_{12}^5+6x_{24}^5+7x_{12}^6+7x_{24}^6+8x_{12}^7+8x_{24}^7-4 \tag{56}$$

$$\Delta^{-1}\left(el_{14}^{\overline{31}\prime}\right)=x_{13}^1+x_{34}^1+2x_{13}^2+2x_{34}^2+4x_{13}^3+4x_{34}^3+5x_{13}^4+5x_{34}^4+6x_{13}^5+6x_{34}^5+7x_{13}^6+7x_{34}^6+8x_{13}^7+8x_{34}^7-4 \tag{57}$$

$$\Delta^{-1}\left(el_{23}^{\overline{12}\prime}\right)=x_{13}^1-x_{12}^1+2x_{13}^2-2x_{12}^2+4x_{13}^3-4x_{12}^3+5x_{13}^4-5x_{12}^4+6x_{13}^5-6x_{12}^5+7x_{13}^6-7x_{12}^6+8x_{13}^7-8x_{12}^7+4 \tag{58}$$

$$\Delta^{-1}\left(el_{23}^{\overline{43}\prime}\right)=x_{24}^1-x_{34}^1+2x_{24}^2-2x_{34}^2+4x_{24}^3-4x_{34}^3+5x_{24}^4-5x_{34}^4+6x_{24}^5-6x_{34}^5+7x_{24}^6-7x_{34}^6+8x_{24}^7-8x_{34}^7+4 \tag{59}$$

$$\Delta^{-1}\left(el_{24}^{\overline{12}\prime}\right)=x_{14}^1-x_{12}^1+2x_{14}^2-2x_{12}^2+4x_{14}^3-4x_{12}^3+5x_{14}^4-5x_{12}^4+6x_{14}^5-6x_{12}^5+7x_{14}^6-7x_{12}^6+8x_{14}^7-8x_{12}^7+4 \tag{60}$$

$$\Delta^{-1}\left(el_{24}^{\overline{31}\prime}\right)=x_{23}^1+x_{34}^1+2x_{23}^2+2x_{34}^2+4x_{23}^3+4x_{34}^3+5x_{23}^4+5x_{34}^4+6x_{23}^5+6x_{34}^5+7x_{23}^6+7x_{34}^6+8x_{23}^7+8x_{34}^7-4 \tag{61}$$

$$\Delta^{-1}\left(el_{34}^{\overline{12}\prime}\right)=x_{14}^1-x_{13}^1+2x_{14}^2-2x_{13}^2+4x_{14}^3-4x_{13}^3+5x_{14}^4-5x_{13}^4+6x_{14}^5-6x_{13}^5+7x_{14}^6-7x_{13}^6+8x_{14}^7-8x_{13}^7+4 \tag{62}$$

$$\Delta^{-1}\left(el_{34}^{\overline{22}\prime}\right)=x_{24}^1-x_{23}^1+2x_{24}^2-2x_{23}^2+4x_{24}^3-4x_{23}^3+5x_{24}^4-5x_{23}^4+6x_{24}^5-6x_{23}^5+7x_{24}^6-7x_{23}^6+8x_{24}^7-8x_{23}^7+4 \tag{63}$$

$$\varepsilon\overline{l_{12}}=\frac{1}{8}\left|\frac{\Delta^{-1}\left(el_{12}^{\overline{33}\prime}\right)+\Delta^{-1}\left(el_{12}^{\overline{43}\prime}\right)}{2}-(x_{12}^1+2x_{12}^2+4x_{12}^3+5x_{12}^4+6x_{12}^5+7x_{12}^6+8x_{12}^7)\right| \tag{64}$$

$$\varepsilon\overline{l_{13}}=\frac{1}{8}\left|\frac{\Delta^{-1}\left(el_{13}^{\overline{21}\prime}\right)+\Delta^{-1}\left(el_{13}^{\overline{43}\prime}\right)}{2}-(x_{13}^1+2x_{13}^2+4x_{13}^3+5x_{13}^4+6x_{13}^5+7x_{13}^6+8x_{13}^7)\right| \tag{65}$$

$$\varepsilon\overline{l_{14}}=\frac{1}{8}\left|\frac{\Delta^{-1}\left(el_{14}^{\overline{21}\prime}\right)+\Delta^{-1}\left(el_{14}^{\overline{31}\prime}\right)}{2}-(x_{14}^1+2x_{14}^2+4x_{14}^3+5x_{14}^4+6x_{14}^5+7x_{14}^6+8x_{14}^7)\right| \tag{66}$$

$$\varepsilon\overline{l_{23}}=\frac{1}{8}\left|\frac{\Delta^{-1}\left(el_{23}^{\overline{12}\prime}\right)+\Delta^{-1}\left(el_{23}^{\overline{43}\prime}\right)}{2}-(x_{23}^1+2x_{23}^2+4x_{23}^3+5x_{23}^4+6x_{23}^5+7x_{23}^6+8x_{23}^7)\right| \tag{67}$$

$$\varepsilon\overline{l_{24}}=\frac{1}{8}\left|\frac{\Delta^{-1}\left(el_{24}^{\overline{12}\prime}\right)+\Delta^{-1}\left(el_{24}^{\overline{31}\prime}\right)}{2}-(x_{24}^1+2x_{24}^2+4x_{24}^3+5x_{24}^4+6x_{24}^5+7x_{24}^6+8x_{24}^7)\right| \tag{68}$$

$$\varepsilon\overline{l_{34}} = \frac{1}{8}\left|\frac{\Delta^{-1}\left(\overline{el_{34}^{12\prime}}\right) + \Delta^{-1}\left(\overline{el_{34}^{22\prime}}\right)}{2} - \left(x_{34}^1 + 2x_{34}^2 + 4x_{34}^3 + 5x_{34}^4 + 6x_{34}^5 + 7x_{34}^6 + 8x_{34}^7\right)\right| \tag{69}$$

$$1 - \frac{\varepsilon\overline{l_{12}} + \varepsilon\overline{l_{13}} + \varepsilon\overline{l_{14}} + \varepsilon\overline{l_{23}} + \varepsilon\overline{l_{24}} + \varepsilon\overline{l_{34}}}{6} \geqslant \overline{CL} \tag{70}$$

Solving the model (49)–(70) based on the Lingo software packages yields the optimal adjusted unbalanced linguistic preference relation $\overline{L}$ and $CL(\overline{L}) = 0.708$.

$$\overline{L} = \begin{pmatrix} - & VH & QH & M \\ - & - & M & QH \\ - & - & - & L \\ - & - & - & - \end{pmatrix}.$$

Moreover, we determine that $\overline{CL} = 0.8$. Then, the optimal adjusted unbalanced linguistic preference relation $\overline{L}$ is as follows:

$$\overline{L} = \begin{pmatrix} - & H & QH & M \\ - & - & M & H \\ - & - & - & VL \\ - & - & - & - \end{pmatrix},$$

and $CL(\overline{L}) = 0.8125$.

Furthermore, we determine that $\overline{CL} = 0.9$. Then, the optimal adjusted unbalanced linguistic preference relation $\overline{L}$ is as follows:

$$\overline{L} = \begin{pmatrix} - & VH & QH & M \\ - & - & M & L \\ - & - & - & VL \\ - & - & - & - \end{pmatrix},$$

and $CL(\overline{L}) = 0.917$.

Moreover, in order to further demonstrate the approach we propose, we will use the lower triangular of the unbalanced linguistic preference relations presented in Cabrerizo et al. [5], and denote it as R.

$$R = \begin{pmatrix} - & - & - & - \\ VL & - & - & - \\ L & M & - & - \\ M & L & QH & - \end{pmatrix}$$

Similar to Section 4.1, we can obtain the consistency level of R, i.e., $CL(R) = 0.708$.

If we determine that $\overline{CL} = 0.8$, then the optimal adjusted unbalanced linguistic preference relation $\overline{R}$ is as follows:

$$\overline{R} = \begin{pmatrix} - & - & - & - \\ VL & - & - & - \\ L & M & - & - \\ M & M & QH & - \end{pmatrix}$$

and $CL(\overline{R}) = 0.833$.

If we determine that $\overline{CL} = 0.9$, then the optimal adjusted unbalanced linguistic preference relation $\overline{R}$ is as follows:

$$\overline{R} = \begin{pmatrix} - & - & - & - \\ L & - & - & - \\ L & M & - & - \\ M & H & QH & - \end{pmatrix}$$

and $CL(\overline{R}) = 0.9375$.

5. Conclusions

Based on the unbalanced linguistic additive transitivity, we propose an optimization-based approach to obtaining a modified unbalanced linguistic preference relation with a required consistency level. The proposed model has several desired advantages.

(i) It preserves the utmost original preference information in the process of improving the consistency, according to the required consistency level.
(ii) It guarantees that the elements in the optimal adjusted unbalanced linguistic preference relation are all simple unbalanced linguistic terms, which accords with the expression of the decision makers.
(iii) The optimum solution to this proposed model can be obtained by a mixed 0–1 linear programming. According to the studies of Miller [29], this mixed 0–1 linear programming is a small-scale optimization problem and can be effectively and rapidly solved by several software packages (e.g., Lingo).

Furthermore, an unbalanced linguistic term set is the extension of a linguistic term set whose terms are uniformly and symmetrically distributed. Therefore, our proposed model can also be applied to improve the consistency level in balanced linguistic preference relations. Moreover, we argue that the optimum solution obtained by mixed 0–1 linear programming should only be considered as a decision aid which decision makers can use as a reference to modify their opinions.

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