A consensus model for hesitant fuzzy preference relations and its application in water allocation management

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A B S T R A C T

This paper investigates a consensus model for hesitant fuzzy preference relations (HFPRs). First, we present a revised definition of HFPRs, in which the values are not ordered for the hesitant fuzzy element. Second, we propose an additive consistency based estimation measure to normalize the HFPRs, based on which, a consensus model is developed. Here, two feedback mechanisms are proposed, namely, interactive mechanism and automatic mechanism, to obtain a solution with desired consistency and consensus levels. In the interactive mechanism, the experts are suggested to give their new preference values in a specific range. If the experts are unwilling to offer their updated preferences, the automatic mechanism could be adopted to carry out the consensus process. Induced ordered weighted averaging (IOWA) operator is used to aggregate the individual HFPRs into a collective one. A score HFPR is proposed for collective HFPR, and then the quantifier-guided dominance degrees of alternatives by using an OWA operator are obtained to rank the alternatives. Finally, both a case of study for water allocation management in Jiangxi Province of China and a comparison with the existing approaches are carried out to show the advantages of the proposed method.

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1. Introduction

Group decision making (GDM) problems mainly aim to obtain the best alternatives from a set of given ones according to the preferences values expressed by a group of experts. Usually, a consensus process and a selection process are carried out before obtaining the final result [4]. Ideally, consensus refers to unanimity among individuals, but it is difficult to accomplish, and "soft" consensus is adopted in practical consensus processes [8]. Generally, the experts talk and change their opinions, sometimes with a moderator, who computes the distances among their opinions. If their opinions differ greatly, the moderator advises them how to update their opinions in order to increase the agreement level.

Consensus has received a great attention in the last few decades [22]. Till now, numerous consensus models and methodologies have been proposed for different types of preference relations: additive reciprocal preference relations [4,5,15,21,28,29,38,44,49], linguistic preference relations [1,3,8,12,25,46], multiplicative preference relations [13,39], and heterogeneous preference representation structures [9,11,14,24]. In addition, consensus models have also been applied in multi-attribute decision making contexts [2,16,48].

In the above cases, there is only one value in the pairwise comparison of all the preference relations. However, it cannot address the situation in which experts are hesitant among a set of possible values. To take into account this situation, Torra [36] first introduced hesitant fuzzy set (HFS) to accommodate the membership degree of an element which has several possible values. The HFS is an extension of a fuzzy set [54]. Rodríguez et al. [31] extended the hesitant fuzzy sets to a linguistic environment, and introduced the hesitant fuzzy linguistic term set (HFLTS). Motivated by the concept of hesitant fuzzy set, and using Saaty's 1/9-9 scale [33] to denote the preference degrees, Xia and Xu [41] introduced the hesitant multiplicative set (HMS). Based on HFS, HFLTS, and HMS, hesitant fuzzy preference relation (HFPR) [41,59,60], incomplete HFPRs [43], hesitant fuzzy linguistic preference relation (HFLPR) [37,58], and hesitant multiplicative preference relation (HMPR) [57] were proposed, respectively. Thereafter, multi-attribute decision making problems were studied. Xia and Xu [40]

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developed several hesitant fuzzy aggregation operators and their applications. Rodríguez et al. [30] developed a novel GDM model based on HFLTSs. Xu et al. [47] proposed a new method called hesitant fuzzy linguistic LINear programming technique for Multi-dimensional Analysis of Preference (LINMAP).

Consensus problems based on hesitant preference relations have also been investigated. Zhang et al. [55] developed a decision support model for GDM based on HFPRs, which not only addresses the consistency, but consensus. The model was also extended to HMPRs by Zhang and Wu [56]. Dong et al. [10] proposed a minimum adjustment consensus model for hesitant linguistic GDM. Xu et al. [45] proposed a dynamic way to adjust weights of DMs automatically in which the DMs express their judgment information by HFPRs. However, there are some limitations of the existing methods. First, the values in the HFSs are generally arranged in ascending or descending order, which may distort the experts' original information. Second, since the numbers of values in different pairwise comparisons are generally not identical, in order to operate correctly, a normalization process, such as β-normalization, is carried out, in which some additional values are added into the original set. However, the added values are random, and the normalization processes are artificial. Finally, in order to obtain the final resolution, there always exists an aggregation process and aggregation operators of hesitant information are used, but the results of these operators are often the combination of all the values in the hesitant fuzzy elements (HFEs) and, consequently, the number of the values in the HFE is increased dramatically, which may cause an increment of the computational complexity.

In order to overcome the previously mentioned drawbacks and limitations, in this paper, we first redefine the concept of the HFPRs. Then, we propose an additive consistency based estimation procedure to normalize the HFPR. Afterwards, we develop a consensus model for GDM problems with HFPRs, which uses both consensus and consistency degrees. Two feedback mechanisms, an interactive mechanism and an automatic mechanism, are proposed to update the experts' hesitant fuzzy preference values. Moreover, we introduce an additive consistency based induced hesitant fuzzy ordered weighted averaging (AC-HFOWA) operator to aggregate all the experts' HFPRs into a collective HFPR. A score HFPR is proposed for collective HFPR, and then the quantifier-guided dominance degrees of alternatives by using an OWA operator are obtained to rank the alternatives.

Water allocation management is related to some objectives and activities with complicated supply-demand conflicts. Competitive water allocation issues among various end-users have become a major topic in research [17]. In particular, such issues emerge in the developing areas of China where multiple water users exist along with relatively low water availability and stability. Therefore, it is urgent for stakeholders to understand how to allocate the limited water resources to satisfy different demands in a cost-effective and environment-friendly manner [20]. Water allocation problem is often characterized by a set of alternatives, complex interactions, and participation of multiple stakeholders with conflict interests. Multi attribute decision making (MADM) is an alternative approach to deal with the water allocation problem, as it facilitates stakeholder's participation and collaborative decision making. The MADM process generally includes the following steps: (1) identifying experts; (2) selecting criteria; (3) formulating alternatives; (4) selecting a MADM technique; (5) assessing the performance of alternatives against the criteria; (6) applying the selected MADM technique; (7) making the final decision [26]. Recently, the use of MADM to solve water resources problems has received considerable attention [18,19,27,34]. In this paper, we develop a consensus model for HFPRs and apply the model to a case study of the water allocation problem.

The remainder of the paper is organized in the following way. Section 2 introduces some basic knowledge of fuzzy preference relations and a revised definition of HFPRs. Section 3 presents a new normalization method for HFPRs which is based on additive consistency, and then the consistency measures for HFPRs are provided. Section 4 develops a consensus model for GDM with HFPRs. Section 5 illustrates a case of study for water allocation management in Jiangxi Province of China. Comparative analyses with the known approaches are also provided to show the advantages of the developed consensus approach. Finally, Section 6 points out some conclusions.

2. Preliminaries

In this section, we introduce some basic knowledge on the definitions of hesitant fuzzy sets, HFPRs, and fuzzy preference relations (FPRs). For simplicity, we denote $N = \{1, 2, ..., n\}$.

Definition 1 ([36]). Let X be a fixed set, a HFS on X is in terms of a function h that returns a non-empty subset of values in [0,1].

For the sake of ease of understanding, HFS can be expressed as ([40]):

$$E = \{< x, h_E(x) > | x \in X\} \tag{1}$$

where $h_E(x)$ is a set of values in [0,1], which denotes the possible membership degrees of the element $x \in X$ to the set E. For convenience, $h_E(x)$ is called a HFE ([40]).

Rodríguez, et al. [32] reviewed more operations, extensions, properties, and applications about HFS.

Definition 2 ([40]). For a HFE h, $\#h$ is the number of the elements in h, then $s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ is the score of h. For two HFEs h_1 and h_2, $h_1 > h_2$ under the condition that $s(h_1) > s(h_2)$; $h_1 = h_2$ under the condition that $s(h_1) = s(h_2)$.

Definition 3 ([35]). An additive FPR R on a finite set of alternatives $X = \{x_1, x_2, ..., x_n\}$ is a fuzzy relation on the product set $X \times X$ with membership function $u_R : X \times X \to [0, 1]$, $u_R(x_i, x_j) = r_{ij}$, verifying

$$r_{ij} + r_{ji} = 1, \ \text{i, j} \in N \tag{2}$$

Usually, a preference relation is represented by an $n \times n$ matrix $R = (r_{ij})_{n \times n}$, in which r_{ij} represents the preference degree of x_i over x_j. Specifically, $0 \leq r_{ij} < 0.5$ denotes a definite preference of x_j over x_i. In particular, $r_{ij} = 0$ denotes that x_j is totally preferred to x_i, and $r_{ij} = 0.5$ denotes x_i and x_j are equally important.

Definition 4 ([35]). A FPR R is called an additively consistent reciprocal preference relation, which verifies:

$$r_{ij} = r_{ik} + r_{kj} - 0.5, \text{for all} i, j \in N \tag{3}$$

In a GDM problem, several experts are involved, and they are invited to evaluate the preference degree to which x_i is preferred to x_j. Some of them offer r_{ij}^1, some offer r_{ij}^2, and the left experts offer r_{ij}^3 ($r_{ij}^1, r_{ij}^2, r_{ij}^3 \in [0, 1]$). In this situation, r_{ij} denotes x_i is preferred to x_j,

containing three values $r_{ij}^1, r_{ij}^2, r_{ij}^3$, and these three values can be regarded as a HFE $r_{ij} = \{r_{ij}^1, r_{ij}^2, r_{ij}^3\}$. Similarly, for the alternatives x_i and x_k, the preference value r_{ik} may be represented as another HFE $r_{ik} = \{r_{ik}^1, r_{ik}^2\}$. The different pairwise comparison r_{ij} contains different number of values, and all r_{ij} can construct a HFPR as follows:

Definition 5. Let $X = \{x_1, x_2, ..., x_n\}$ be an alternative set. A HFPR H on X is denoted by a matrix $H = (h_{ij})_{n \times n} \subset X \times X$, where $h_{ij} = \{h_{ij}^\beta | \beta = 1, 2, ..., \#h_{ij}\}$ ($\#h_{ij}$ is the number of elements in h_{ij}) is a HFE, which denotes all the possible preference degree(s) of the alternative x_i over x_j. Additionally, h_{ij} should verify:

$$h_{ij}^\beta + h_{ji}^\beta = 1, \ h_{ii} = \{0.5\}, \ \#h_{ij} = \#h_{ji}, \ i, j = 1, 2, ..., n \tag{4}$$

where h_{ij}^β is the βth value in h_{ij}.

Remark 1. Xia and Xu [41] first extended the concept of FPR by using HFS to introduce the definition of HFPR. However, Definition 4 is a little different to Xia and Xu [41]'s definition. In Xia and Xu [41]'s definition of HFPR, it needs the values in h_{ij} are supposed to be arranged in ascending order, i.e., $h_{ij}^\beta < h_{ij}^{\beta+1}$ ($i < j$). Definition 4 does not have the constraint, which does not rearrange the elements in ascending or descending order. For example, when an expert gives his/her comparison information on alternatives x_1 and x_2, he/she gives two membership degree 0.6, and 0.9, which denotes that he/she thinks x_1 is moderately preferred to x_2, and x_1 is extremely preferred to x_2. Moreover, he/she is sure of the membership degree 0.9 than 0.6. In this situation, the comparison information for x_1 and x_2 given by this expert denoted by HFS should be {0.9, 0.6}. If we order the values in ascending order, it will be {0.6, 0.9}, which will distort the expert's original information. Furthermore, as we will show in the following, it is not appropriate and may have some problems in the consistent process.

3. Consistency of HFPRs

In this section, a new normalization method for HFPRs is presented. It is based on additive consistency.

3.1. Normalization of HFPRs

Given any two HFEs h_1 and h_2, in most cases, $\#h_1 \neq \#h_2$. As the numbers of different pairwise comparison elements in a hesitant preference relation may be not identical, in order to operate correctly, Zhu et al. [60] proposed a method called β-normalization to make that all the comparisons have the same number of elements.

Definition 6 ([60]). Let $h = \{h^\beta | \beta = 1, 2, ..., \#h\}$ be a HFE, h^+ and h^- be the maximum and minimum elements in h, respectively, and ς ($0 \leq \varsigma \leq 1$) be a parameter, then $\bar{h} = \varsigma h^+ + (1 - \varsigma)h^-$ is called an added element.

Remark 2. Particularly, if $\varsigma = 1$, then $\bar{h} = h^+$, and $\bar{h} = h^-$ when $\varsigma = 0$, which Xu and Xia [50] called them optimism and pessimism rules, respectively. Although Xu and Xia [50]'s and Zhu et al. [60]'s methods can make all the comparisons have the same number of elements. However, there are some limitations in their methods. In Xu and Xia [50]'s method, the added value is only the minimum or maximum value in the set, and other medium values cannot be the added values. For example, for a given HFE $h = \{0.1, 0.2, 0.3\}$, $h = \{0.1, 0.1, 0.2, 0.3\}$ and $h = \{0.1, 0.2, 0.3, 0.3\}$ are only two possible extended HFEs which has a length of 4 HFEs. In Zhu et al. [60]'s method, numerous possible added elements can be obtained. For the former example, if $\varsigma = 0.3$, we will have one of the extended length 4 HFE $h = \{0.1, 0.16, 0.2, 0.3\}$. In our opinion, the added value 0.16 is not the original value, and there is no rule that show how to set the parameter ς, which will distort the experts' original information. The normalization process is artificial and random. Furthermore, for the HFE $h = \{0.1, 0.2, 0.3\}$, the original HFE may be $h = \{0.1, 0.2, 0.2, 0.3\}$, in which two experts offer the same values 0.2, while we denote these values as a HFE, the multi-values only appear once. In our opinion, the added values should be based on some rules. Due to the limitations of Xu and Xia [50]'s and Zhu et al. [60]'s methods, in the following, we propose another method to add values.

Definition 7. Let $h = \{h^\beta | \beta = 1, 2, ..., \#h\}$ be a HFE, $\bar{h}$ be the direct added value, which satisfies $\bar{h} \in h$, and the elements are added after the existing values.

Remark 3. Definition 7 shows that the direct added elements should be the elements in the original HFE which aims to surmise the expert's original information as much as possible. Furthermore, the added values should be put after the existing values. For example, given a HFE $h = \{0.1, 0.2, 0.3\}$, if 0.2 is the added element, then $h = \{0.1, 0.2, 0.3, 0.2\}$. All the above three methods can add the elements in such a way that all the comparisons have the same number of elements. In the following, we propose another method called additive consistency-based method to add the elements.

In the consistency-based method, we look the elements to be added as unknown values [6,23]. According to the additive consistency property (Eq. (3)) of the FPR, we develop an algorithm to estimate the unknown values. By using the additive consistency (i.e., $r_{ij} = r_{ik} + r_{kj} - 0.5$) of $R = (r_{ij})_{n \times n}$, and the intermediate alternative x_k, we can estimate the preference value r_{ij} in the following three ways [21,23]:

1) From $r_{ij} = r_{ik} + r_{kj} - 0.5$, we obtain the estimation

$$cr_{ij}^{k1} = r_{ik} + r_{kj} - 0.5 \tag{5}$$

2) From $r_{kj} = r_{ki} + r_{ij} - 0.5$, we obtain the estimation

$$cr_{ij}^{k2} = r_{kj} - r_{ki} + 0.5 \tag{6}$$

3) From $r_{ik} = r_{ij} + r_{jk} - 0.5$, we obtain the estimation

$$cr_{ij}^{k3} = r_{ik} - r_{jk} + 0.5 \qquad (7)$$

If the reciprocal property of a FPR is considered, we conclude that $cr_{ij}^{k1} = cr_{ij}^{k2} = cr_{ij}^{k3}$ in (5)–(7), because

$$cr_{ij}^{k2} = r_{kj} - r_{ki} + 0.5 = r_{kj} - (1 - r_{ik}) + 0.5 = r_{ik} + r_{kj} - 0.5 = cr_{ij}^{k1};$$

$$cr_{ij}^{k3} = r_{ik} - r_{jk} + 0.5 = r_{ik} - (1 - r_{kj}) + 0.5 = r_{ik} + r_{kj} - 0.5 = cr_{ij}^{k1}.$$

Therefore, we can utilize any one of (5)–(7) to estimate r_{ij}. For simplicity, we set $cr_{ij}^k = cr_{ij}^{k1}$. Then, we can obtain the overall estimated value cr_{ij} of r_{ij} by

$$cr_{ij} = \frac{\sum_{k=1}^n cr_{ij}^k}{n} \qquad (8)$$

Here, we allow $i = j = k$, since the additive consistency property (Eq. (3)) holds for all $i, j, k \in N$.

For an incomplete FPR R, we introduce the following sets [23]:

$$A = \{(i, j) | i, j \in N \wedge i \neq j\}$$

$$MV = \{(i, j) \in A | r_{ij} \text{ is missing}\}$$

$$EV = A \backslash MV$$

$$P_{ij} = \{k \neq i, j | (i, k), (k, j) \in EV\}$$

MV is the set of unknown values. EV is the set of pairs of known values. P_{ij} is the set of x_k which can be used for the estimation of r_{ij} ($i \neq j$) by Eq. (5).

In the following, we develop a revised algorithm to estimate the unknown values for incomplete FPRs. The procedure includes the two steps.

(1) Establish the elements that can be estimated in each iteration

Let R be an incomplete FPR, EMV_t be the subset of unknown values MV, which can be estimated in iteration t and is defined as:

$$EMV_t = \left\{ (i, j) \in MV \backslash \bigcup_{l=0}^{t-1} EMV_l | i \neq j \wedge \exists k \in \{P_{ij}\} \right\} \qquad (9)$$

Obviously, $EMV_0 = \emptyset$ the procedure will stop when $EMV_{maxIter} = \emptyset$. In this case, there are no more unknown values which can be estimated. Furthermore, if $\bigcup_{l=1}^{maxIter} EMV_l = MV$, then all unknown values have been estimated, in this case, we have successfully achieved the completion for the incomplete FPR.

(2) Estimation of a given unknown value

To estimate a given unknown value r_{ij} with $(i, j) \in EMV_t$, we establish the following function estimate_$r(i,j)$:

$$\text{function estimate\_}r(i,j)$$

$$\text{if} \quad \# P_{ij} \neq 0 \quad \text{then}$$

$$cr_{ij} = \frac{\sum_{k \in P_{ij}} cr_{ij}^k}{\# P_{ij}}$$

$$\text{end function}$$

The function estimate_$r(i,j)$ calculates the average of all the estimate values using all the possible intermediate alternative x_k using Eq. (5) as the final estimated value cr_{ij}. However, the estimated values cr_{ij} may be out of the unit interval [0,1]. To avoid such unnormalized values, we define cr_{ij} as follows:

$$cr_{ij} = med\{0, 1, cr_{ij}\} \qquad (10)$$

The iterative estimation procedure is:

> 0. $EMV_0 = \varnothing$
>
> 1. $t = 1$
>
> 2. while $EMV_t \neq \varnothing$ {
>
> 3.   for every $(i, j) \in EMV_t$ {
>
> 4.   Estimate_$r(i,j)$
>
> 5.   }
>
> 6. $t++$
>
> 7. }

Definition 8. Let $H = (h_{ij})_{n \times n}$ be a HFPR, a Normalized Hesitant Fuzzy Preference Relation (NHFPR) $\bar{H} = (\bar{h}_{ij})_{n \times n}$ should satisfy the following condition

$$\# \bar{h}_{ij} = \max\{\# \bar{h}_{ij} | i, j = 1, 2, ..., n\} \quad (i, j, = 1, 2, ..., n; \, i \neq j)$$

$$\bar{h}_{ij}^{\beta} + \bar{h}_{ji}^{\beta} = 1, \, \bar{h}_{ii} = 0.5,$$

where $\bar{h}_{ij}^{\beta}$ is the βth element in $\bar{h}_{ij}$.

Remark 4. The above definition of NHFPR is slightly different from Zhu et al. [60]'s, as Zhu et al. [60] used Definition 3 to add the elements randomly. In addition, their method needs the values in h_{ij} of the upper triangular are rearranged in ascending order, i.e., $\bar{h}_{ij}^{\beta} < \bar{h}_{ij}^{\beta+1}$ $(i < j)$. Here, we adopt the additive consistency based method to add the elements, which is more reasonable than Zhu et al. [60]'s method.

Example 1. Let H_1 be a HFPR, which is shown as follows:

$$H_1 = \begin{pmatrix} \{0.5\} & \{0.6, 0.7\} & \{0.2, 0.3\} & \{0.4\} \\ \{0.4, 0.3\} & \{0.5\} & \{0.1, 0.2\} & \{0.8, 0.9\} \\ \{0.8, 0.7\} & \{0.9, 0.8\} & \{0.5\} & \{0.7, 0.8\} \\ \{0.6\} & \{0.2, 0.1\} & \{0.3, 0.2\} & \{0.5\} \end{pmatrix}$$

First, we transform H_1 into two FPRs:

$$R_1^{H_1} = \begin{pmatrix} 0.5 & 0.6 & 0.2 & 0.4 \\ 0.4 & 0.5 & 0.1 & 0.8 \\ 0.8 & 0.9 & 0.5 & 0.7 \\ 0.6 & 0.2 & 0.3 & 0.5 \end{pmatrix}, \, R_2^{H_1} = \begin{pmatrix} 0.5 & 0.7 & 0.3 & x \\ 0.3 & 0.5 & 0.2 & 0.9 \\ 0.7 & 0.8 & 0.5 & 0.8 \\ x & 0.1 & 0.2 & 0.5 \end{pmatrix}.$$

Obviously, $R_2^{H_1}$ is an incomplete FPR. We apply the consistency-based procedure to estimate the unknown values. Iteration 1) The set of estimation values is:

$$EMV_1 = \{(1, 4), (4, 1)\}$$

To estimate r_{14} of $R_2^{H_1}$, the procedure is as follows:

$$P_{14} = \{2, 3\} \Rightarrow cr_{14}^2 = r_{12} + r_{24} - 0.5 = 0.7 + 0.9 - 0.5 = 1.1$$

$$cr_{14}^3 = r_{13} + r_{34} - 0.5 = 0.3 + 0.8 - 0.5 = 0.6$$

$$cr_{14} = \frac{1.1 + 0.6}{2} = 0.85$$

Similarly, we have $cr_{14} = 0.15$.
After the estimation of all these unknown values, we obtain the following complete FPR $R_2^{H_1}$.

$$R_2^{H_1} = \begin{pmatrix} 0.5 & 0.7 & 0.3 & 0.85 \\ 0.3 & 0.5 & 0.2 & 0.9 \\ 0.7 & 0.8 & 0.5 & 0.8 \\ 0.15 & 0.1 & 0.2 & 0.5 \end{pmatrix}$$

Then, the NHFPR $\bar{H}_1$ is:

$$\bar{H}_1 = \begin{pmatrix} \{0.5\} & \{0.6, 0.7\} & \{0.2, 0.3\} & \{0.4, 0.85\} \\ \{0.4, 0.3\} & \{0.5\} & \{0.1, 0.2\} & \{0.8, 0.9\} \\ \{0.8, 0.7\} & \{0.9, 0.8\} & \{0.5\} & \{0.7, 0.8\} \\ \{0.6, 0.15\} & \{0.2, 0.1\} & \{0.3, 0.2\} & \{0.5\} \end{pmatrix}$$

If the expert does not find any comparison information for h_{23} of H_1, then H_1 is an incomplete HFPR as follows:

$$H_1' = \begin{pmatrix} \{0.5\} & \{0.6, 0.7\} & \{0.2, 0.3\} & \{0.4\} \\ \{0.4, 0.3\} & \{0.5\} & x & \{0.8, 0.9\} \\ \{0.8, 0.7\} & x & \{0.5\} & \{0.7, 0.8\} \\ \{0.6\} & \{0.2, 0.1\} & \{0.3, 0.2\} & \{0.5\} \end{pmatrix}$$

First, we transform H_1' into the following two incomplete FPR:

$$R_1^{H_1'} = \begin{pmatrix} 0.5 & 0.6 & 0.2 & 0.4 \\ 0.4 & 0.5 & x & 0.8 \\ 0.8 & x & 0.5 & 0.7 \\ 0.6 & 0.2 & 0.3 & 0.5 \end{pmatrix}, R_2^{H_1'} = \begin{pmatrix} 0.5 & 0.7 & 0.3 & x \\ 0.3 & 0.5 & x & 0.9 \\ 0.7 & x & 0.5 & 0.8 \\ x & 0.1 & 0.2 & 0.5 \end{pmatrix}$$

Applying the additive consistency-based estimation procedure, we have:

$$R_1^{H_1'} = \begin{pmatrix} 0.5 & 0.6 & 0.2 & 0.4 \\ 0.4 & 0.5 & 0.55 & 0.8 \\ 0.8 & 0.45 & 0.5 & 0.7 \\ 0.6 & 0.2 & 0.3 & 0.5 \end{pmatrix}, R_2^{H_1'} = \begin{pmatrix} 0.5 & 0.7 & 0.3 & 0.85 \\ 0.3 & 0.5 & 0.35 & 0.9 \\ 0.7 & 0.65 & 0.5 & 0.8 \\ 0.15 & 0.1 & 0.2 & 0.5 \end{pmatrix}$$

Then, the NHFPR H_1' is:

$$\bar{H}_1' = \begin{pmatrix} \{0.5\} & \{0.6, 0.7\} & \{0.2, 0.3\} & \{0.4, 0.85\} \\ \{0.4, 0.3\} & \{0.5\} & \{0.55, 0.35\} & \{0.8, 0.9\} \\ \{0.8, 0.7\} & \{0.45, 0.65\} & \{0.5\} & \{0.7, 0.8\} \\ \{0.6, 0.15\} & \{0.2, 0.1\} & \{0.3, 0.2\} & \{0.5\} \end{pmatrix}$$

Example 2. Let H_2 be a HFPR, which is shown as follows:

$$H_2 = \begin{pmatrix} \{0.5\} & \{0.4, 0.6, 0.7\} & \{0.2, 0.3\} & \{0.5, 0.7\} \\ \{0.6, 0.4, 0.3\} & \{0.5\} & \{0.3, 0.4\} & \{0.8, 0.9\} \\ \{0.8, 0.7\} & \{0.7, 0.6\} & \{0.5\} & \{0.1\} \\ \{0.5, 0.3\} & \{0.2, 0.1\} & \{0.9\} & \{0.5\} \end{pmatrix}$$

First, we transform H_2 into three FPRs:

$$R_1^{H_2} = \begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.5 \\ 0.6 & 0.5 & 0.3 & 0.8 \\ 0.8 & 0.7 & 0.5 & 0.1 \\ 0.5 & 0.2 & 0.9 & 0.5 \end{pmatrix}, R_2^{H_2} = \begin{pmatrix} 0.5 & 0.6 & 0.3 & 0.7 \\ 0.4 & 0.5 & 0.4 & 0.9 \\ 0.7 & 0.6 & 0.5 & x \\ 0.3 & 0.1 & x & 0.5 \end{pmatrix}, R_3^{H_2} = \begin{pmatrix} 0.5 & 0.7 & x & x \\ 0.3 & 0.5 & x & x \\ x & x & 0.5 & x \\ x & x & x & 0.5 \end{pmatrix}$$

In this example, $R_2^{H_2}$ and $R_3^{H_2}$ are incomplete FPRs. Applying the estimation procedure which is based on additive consistency, $R_2^{H_2}$ can be completed as:

$$R_2^{H_2} = \begin{pmatrix} 0.5 & 0.6 & 0.3 & 0.7 \\ 0.4 & 0.5 & 0.4 & 0.9 \\ 0.7 & 0.6 & 0.5 & 0.95 \\ 0.3 & 0.1 & 0.05 & 0.5 \end{pmatrix}$$

$R_3^{H_2}$ is not an acceptable incomplete FPR, thus, we first use Definition 4 to add some elements until each row and column has at least one known element. For instance:

$$R_3^{H_2} = \begin{pmatrix} 0.5 & 0.7 & x & x \\ 0.3 & 0.5 & 0.4 & 0.9 \\ x & 0.6 & 0.5 & x \\ x & 0.1 & x & 0.5 \end{pmatrix}$$

Then, applying the consistency-based estimation procedure, we obtain

$$R_3^{H_2} = \begin{pmatrix} 0.5 & 0.7 & 0.6 & 1 \\ 0.3 & 0.5 & 0.4 & 0.9 \\ 0.4 & 0.6 & 0.5 & 1 \\ 0 & 0.1 & 0 & 0.5 \end{pmatrix}$$

Therefore, the NHFPR $\bar{H}_2$ is:

$$\bar{H}_2 = \begin{pmatrix} \{0.5\} & \{0.4, 0.6, 0.7\} & \{0.2, 0.3, 0.6\} & \{0.5, 0.7, 1\} \\ \{0.6, 0.4, 0.3\} & \{0.5\} & \{0.3, 0.4, 0.4\} & \{0.8, 0.9, 1\} \\ \{0.8, 0.7, 0.4\} & \{0.7, 0.6, 0.6\} & \{0.5\} & \{0.1, 0.95, 1\} \\ \{0.5, 0.3, 0\} & \{0.2, 0.1, 0\} & \{0.9, 0.05, 0\} & \{0.5\} \end{pmatrix}$$

Remark 5. From Examples 1 and 2, using Definition 7 and the consistency-based estimation procedure, we can obtain the NHFPR, which tries to surmise the expert's original preference values, and avoid adding the values randomly. Furthermore, the proposed method can be used to normalize the incomplete HFPR, while Xu and Xia [50]'s, Zhu et al. [60]'s and Zhang et al. [55]' methods only can be used for complete HFPR.

3.2. Consistency measure of HFPR

Definition 9. Let $H = (h_{ij})_{n \times n}$ be a HFPR and $\bar{H} = (\bar{h}_{ij})_{n \times n}$ its NHFPR, for all $i, j \in N$, if

$$\bar{h}_{ik}^{\beta} + \bar{h}_{kj}^{\beta} = \bar{h}_{ij}^{\beta} + 0.5 \tag{11}$$

being $\bar{h}_{ij}^{\beta}$ the βth value in $\bar{h}_{ij}$, then H is said to be an additively consistent HFPR.

Based on Eq. (5), we can compute estimated $CH = (ch_{ij})$ ($ch_{ij} = \{ch_{ij}^{\beta} | \beta = 1, 2, ..., \#ch_{ij}\}$ as follows:

$$\left(ch_{ij}^{\beta} \right)^k = \bar{h}_{ik}^{\beta} + \bar{h}_{kj}^{\beta} - 0.5 \tag{12}$$

Then

$$ch_{ij}^{\beta} = \frac{\sum_{k=1}^{n} \left(ch_{ij}^{\beta} \right)^k}{n} = \frac{\sum_{k=1}^{n} \left(\bar{h}_{ik}^{\beta} + \bar{h}_{kj}^{\beta} \right)}{n} - 0.5 \tag{13}$$

When $ch_{ij}^{\beta} = \bar{h}_{ij}^{\beta}$, the HFPR is completely additively consistent. However, Eq. (11) does not always hold. In the following, the deviation between a hesitant fuzzy element and its estimation value can be defined as:

Definition 10. The deviation between a normalized hesitant fuzzy preference element $\bar{h}_{ij}$ and its final estimation ch_{ij} is:

$$\varepsilon h_{ij} = \frac{1}{\#h_{ij}} \sum_{\beta=1}^{\#h_{ij}} |ch_{ij}^{\beta} - \bar{h}_{ij}^{\beta}| \tag{14}$$

As the HFPR is reciprocal (i.e., $h_{ij}^{\beta} + h_{ji}^{\beta} = 1$), which implies $CH = (ch_{ij})$ is also reciprocal (i.e., $ch_{ij}^{\beta} + ch_{ji}^{\beta} = 1$), therefore, $\varepsilon h_{ij} = \varepsilon h_{ji}$. εh_{ij} can be used to measure the error of a HFE for a pair of alternatives. Therefore, it can be denoted as the consistency level associated with the HFE h_{ij} in a HFPR.

Definition 11. The consistency degree associated with h_{ij} is:

$$cl_{ij} = 1 - \varepsilon h_{ij} \tag{15}$$

Definition 12. The consistency index associated with an alternative x_i is defined as

$$cl_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{cl_{ij}}{n-1} \tag{16}$$

Definition 13. The consistency degree of a NHFPR $\bar{H}$ can be measured by:

$$cl = \sum_{i=1}^{n} \frac{cl_i}{n} \tag{17}$$

Obviously $0 \leq cl \leq 1$. If $cl = 1$, it denotes that NHFPR $\bar{H}$ is of complete consistency. On the contrary, the smaller the value of cl, the more inconsistent $\bar{H}$.

Remark 6. In the above, we have provided an additive-consistency based method to normalize the HFPR, and we have developed the consistency degree in three levels. The normalization process can use the users' information sufficiently, and the consistency indexes can show different levels of the consistency, while the Zhang et al. [55]'s method only can know the global consistency.

4. A consensus model for GDM with HFPRs

Suppose that for a set of experts $E = \{e_1, e_2, ..., e_m\}$, each expert offers his/her preference relation. In the process of a GDM problem, we hope to reach a high consensus degree among experts before the final resolution. The experts could change their preference values according to a moderator's recommendation. To solve the GDM problems with HFPRs, first it is necessary to normalize the HFPRs. The previous definition to add the elements and the additively consistent based estimation procedure allow us to normalize the HFPRs to the ones with higher consistency degrees. After normalization process, we measure the consistency degrees for these normalized HFPRs. Furthermore, an additive-consistency (AC) inducing hesitant fuzzy ordered weighted averaging (IHFOWA) operator is proposed to aggregate the individual HFPRs to a collective one. The weighting vector of AC-IHFOWA operator is derived by a linguistic quantifier, in which the expert's additive consistency degree is considered. The higher consistency degree, the more the weight is, and therefore the higher contribution to the collective HFPR. When we get the collective HFPR, a proximity degree (PD) is computed. Both the consistency and consensus levels are utilized to control the consensus reaching process. When the consistency/consensus levels have achieved a predefined threshold, the resolution process is implemented; otherwise, the recommendation processes are executed.

The consensus model with HFPRs is illustrated in Fig. 1. It has the following stages: (1) Normalization of HFPRs; (2) Calculating consistency degrees; (3) Calculating consensus degrees; (4) Proximity degrees; (5) Feedback mechanisms; (6) Aggregation process; (7) Selection process. The first stage has been presented in Section 3. The rest of stages will be addressed in the following subsections.

4.1. Consistency degree calculation

To obtain consistency degrees of each NHFPR $\bar{H}_\tau$, we use Eq. (13) to calculate its consistent NHFPR $CH_\tau = (ch_{ij,\tau})$. Afterwards, we apply Eqs. (14)–(17) to compute the consistency degrees $CL^\tau = (cl_{ij}^\tau)$, cl_i^τ, cl^τ $\forall i, j \in N$. In the end, a global consistency degree of all the experts is defined, which is used to control the global consistency in a predefined threshold.

Definition 14. The global consistency degree is defined as follows:

$$CL = \frac{\sum_{\tau=1}^m cl^\tau}{m} \tag{18}$$

4.2. Consensus degree calculation

We define a similarity matrix $SM^{\mu\tau} = (sm_{ij}^{\mu\tau})$ for experts (e_μ, e_τ) $(\mu < \tau)$, where:

$$sm_{ij}^{\mu\tau} = 1 - \frac{1}{\#\bar{h}_{ij}} \sum_{\beta=1}^{\#\bar{h}_{ij}} |\bar{h}_{ij,\mu}^\beta - \bar{h}_{ij,\tau}^\beta| \tag{19}$$

Obviously, there are $(m-1) \times (m-2)$ similarity matrices. We then use the arithmetic aggregation operator ϕ to aggregate all the similarity matrices and to obtain a collective similarity matrix $SM = (sm_{ij})$ as follows:

$$sm_{ij} = \phi(sm_{ij}^{\mu\tau}), \ \mu, \tau = 1, 2, ..., m, \ i, j = 1, 2, ..., n, \ \mu < \tau \tag{20}$$

Once we obtain the similarity matrix, we compute the consensus degree in the following three different levels:
Level 1. Consensus degree cp_{ij} on pairs of alternatives (x_i, x_j), which measures the agreement among all experts on the pair of alternatives:

$$cp_{ij} = sm_{ij} \tag{21}$$

Level 2. Consensus degree ca_i on alternatives x_i, which is the average of the consensus degrees cp_{ij}:

$$ca_i = \frac{\sum_{j=1, j \neq i}^n cp_{ij}}{n-1} \tag{22}$$

Level 3. Consensus degree CR on the relation SM, which is the average of consensus degrees ca_i:

$$CR = \frac{\sum_{i=1}^n ca_i}{n} \tag{23}$$

4.3. Proximity measures

In order to measure how close the individual preferences are from the group preference, proximity measures are proposed. The group preference values are computed by aggregating all the individual preference values using the IOWA operator. Yager and Filev [53] first introduced the IOWA operator, which is guided by fuzzy linguistic quantifiers. In the IOWA operator, the argument ordering process is guided by a variable called the order inducing value, which allows the introduction of some semantics of meaning in the aggregation, and therefore allows for a better control over the aggregation stage in the resolution process of the GDM problems. In this paper, we propose an Additive Consistency (AC) Induced Hesitant Fuzzy Ordered Weighted (IHFOWA) operator, in which the weight is guided by the additive consistency: the more consistency, the more the weight is, and thus, the collective HFPR can reflect an individual's consistency degree. Therefore, we use the IOWA to aggregate the individual HFPRs.
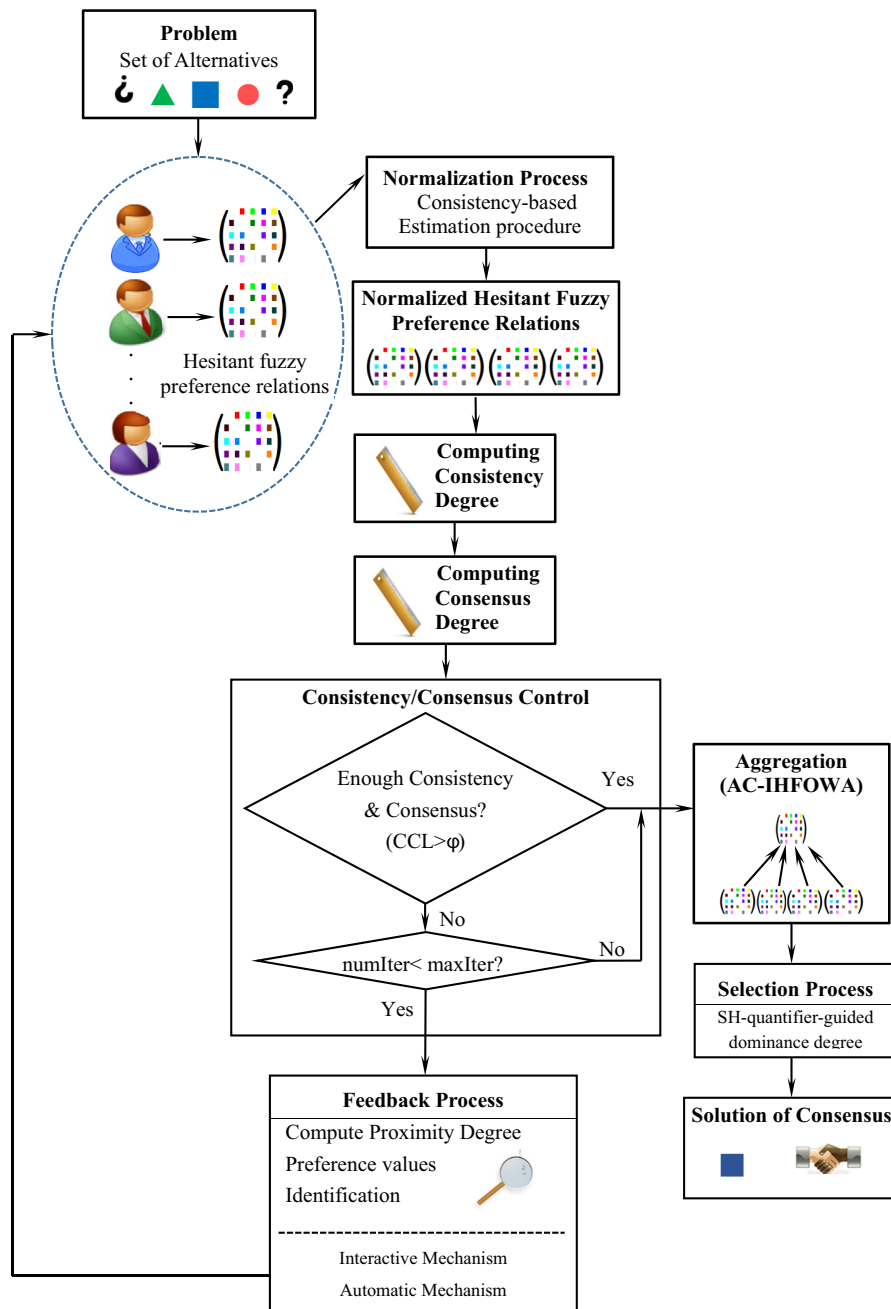
Fig. 1. Consensus model for HFPRs.

Definition 15 *([53])*. An IOWA operator is defined as:

$$\Phi_w(\langle u_1, p_1 \rangle \langle u_2, p_2 \rangle \ldots \langle u_m, p_m \rangle) = \sum_{i=1}^{m} w_i p_{\sigma(i)} \tag{24}$$

where $W = (w_1, w_2, \ldots, w_m)^T$ is a weighting vector, such that $w_i \in [0, 1]$, $\sum_{i=1}^{m} w_i = 1$, σ is a permutation of $\{1, 2, \ldots, m\}$ such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}$, $\forall i = 1, \ldots, m-1$, i.e., $\langle u_{\sigma(i)}, p_{\sigma(i)} \rangle$ is the 2-tuple with $u_{\sigma(i)}$ the ith largest value in the set $\{u_1, \ldots, u_m\}$.

As shown in Definition 15, the IOWA operator takes as its argument pairs, in which the second component $p_{\sigma(i)}$ is reordered according to the first component $u_{\sigma(i)}$. u_i ($i = 1, 2, \ldots, m$) is referred as the order inducing variable and p_i ($i = 1, 2, \ldots, m$) as the argument variable [53].

A normal problem in the IOWA is how to set its associated weighting vector. There are so many ways to obtain it. One of the well-known methods is the use of a linguistic quantifier Q, which was first proposed by Yager [52]:

$$w_\tau = Q \left(\frac{\sum_{k=1}^{\tau} u_{\sigma(k)}}{T} \right) - Q \left(\frac{\sum_{k=1}^{\tau-1} u_{\sigma(k)}}{T} \right) \tag{25}$$

In the above, $T = \sum_{k=1}^{m} u_{\sigma(k)}$ denotes the overall sum of importance.

Based on the IOWA operator, we use the consistency level as the order induced variable, and then we propose an Additive Consistency (AC) Induced Hesitant Fuzzy Ordered Weighted (IHFOWA) operator to obtain the collective HFPR:

$$h_{ij,c}^{(\beta)} = \Phi_W(\langle cl^1, \bar{h}_{ij,1}^{\beta}\rangle, \langle cl^2, \bar{h}_{ij,2}^{\beta}\rangle...\langle cl^m, \bar{h}_{ij,m}^{\beta}\rangle) = \sum_{\tau=1}^{m} w_\tau \bar{h}_{ij,\sigma(\tau)}^{\beta} \tag{26}$$

with $cl^{\sigma(\tau-1)} \geq cl^{\sigma(\tau)}$.

In the AC-IHFOWA operator, the induced variable is the consistency values cl^τ, and then the aggregated values $\bar{h}_{ij,\sigma(\tau)}^{\beta}$ are ordered according to the experts' consistency level cl^τ. Applying Eq. (25), we can obtain the weights of the AC-IHFOWA operator as follows:

$$w_\tau = Q\left(\frac{\sum_{k=1}^{\tau} cl^{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{\tau-1} cl^{\sigma(k)}}{T}\right) \tag{27}$$

with $T = \sum_{k=1}^{m} cl^{\sigma(k)}$, and $cl^{\sigma(k)}$ is the kth largest value of the set $\{cl^1, cl^2, ..., cl^m\}$.

Level 1. Proximity pp_{ij}^τ on pairs of alternatives (x_i, x_j), which measures the similarity between an individual expert's preference value and the corresponding collective one:

$$pp_{ij}^\tau = 1 - \frac{1}{\#\bar{h}_{ij}} \sum_{\beta=1}^{\#\bar{h}_{ij}} |\bar{h}_{ij,\tau}^{\beta} - \bar{h}_{ij,c}^{\beta}| \tag{28}$$

Level 2. Proximity pa_i^τ on alternatives x_i, which is the average of the proximity pp_{ij}^τ of alternative x_i:

$$pa_i^\tau = \frac{\sum_{j=1, j\neq i}^{n} pa_{ij}^\tau}{n-1} \tag{29}$$

Level 3. Proximity pr^τ on the relation $\bar{H}_\tau$, which is the average of the pa_i^τ of expert e_τ:

$$pr^\tau = \frac{\sum_{i=1}^{n} pa_i^\tau}{n} \tag{30}$$

4.4. Computing consistency/consensus levels

In GDM problems, we have to decide when the consensus will be achieved. Generally, the consistency/consensus level should be in a given threshold. In order to realize it, in the following, a satisfaction level, called consistency/consensus level (CCL) is defined:

$$CCL = \lambda \cdot CL + (1 - \lambda) \cdot CR \tag{31}$$

where $\lambda \in [0, 1]$ is a parameter. If $\lambda > 0.5$, more importance is given to the consistency level, while if $\lambda < 0.5$, more importance is given to the consensus level. Generally, a minimum threshold $\varphi \in [0, 1]$ is specified for CCL. If CCL is larger than φ, the consensus is achieved and the selection process is carried out. Furthermore, the iteration process should be executed without excessive rounds, and thus we also predefine a maximum number of iterations.

4.5. Feedback mechanism

Consensus reaching process is always a dynamic and negotiation process, in which there exists a moderator, who helps the experts to modify their opinions in order to make them closer. The moderator knows all the individual and global consistency and consensus levels, if CCL is smaller than a predefined threshold φ, the moderator will generate personalized recommendations for each expert to revise their opinions, which is called feedback mechanism. It has two steps: (1) *The preference values identification*; and (2) *Advice generation*.

4.5.1. The preference values identification

Obviously, to modify the experts' preference values, we should identify the values which contributes less to the consistency/consensus level. Based on the above different levels of proximity measures, we first determine the experts, second the alternatives, and finally the preference values.

Step 1) Identify the experts EXPCH who should change their preference values:

$$EXPCH = \{\tau | \lambda \cdot cl^\tau + (1 - \lambda) \cdot pr^\tau < \varphi\} \tag{32}$$

Step 2) Identify the alternatives ALT for the identified experts, i.e.,

$$ALT = \{(\tau, i) | \tau \in EXPCH \wedge \lambda \cdot cl_i^\tau + (1 - \lambda) \cdot pa_i^\tau < \varphi\} \tag{33}$$

Step 3) Identify preference values APS for the identified alternatives, i.e.,

$$APS = \{(\tau, i, j), (\tau, j, i) | (\tau, i) \in ALT \wedge \lambda \cdot cl_{ij}^\tau + (1 - \lambda) \cdot pp_{ij}^\tau = \lambda \cdot cl_{ji}^\tau + (1 - \lambda) \cdot pp_{ji}^\tau < \varphi\} \tag{34}$$

4.5.2. Advice generation

Once we have identified the preference values, the feedback mechanism is activated to generate advice with the aim of helping the experts to update their personal preference values. In the following, we provide two mechanisms of advice generation. One is an interactive mechanism and the other is an automatic mechanism.

The interactive mechanism offers to the experts the direction in which they have to modify their preferences, which is based on the comparison between the identify preference values APS and collective pReferences

(1) If $\bar{h}_{ij,\tau}^{\beta} - \bar{h}_{ij,c}^{\beta} < 0$, expert e_τ is suggested to increase $\bar{h}_{ij,\tau}^{\beta}$;

(2) If $\bar{h}_{ij,\tau}^{\beta} - \bar{h}_{ij,c}^{\beta} > 0$, expert e_τ is suggested to decrease $\bar{h}_{ij,\tau}^{\beta}$;

(3) If $\bar{h}_{ij,\tau}^{\beta} - \bar{h}_{ij,c}^{\beta} = 0$, expert e_τ is suggested to not change $\bar{h}_{ij,\tau}^{\beta}$.

The above recommendation only provides to the experts the direction to update their preference values, but it does not provide to the experts how much the value should be changed, or the range of the values that can be changed. When the experts provide the new values, we suggest the new value $rh_{ij,\tau}^{\beta}$ is as an interval, i.e. $rh_{ij,\tau}^{\beta} \in [\min\{\bar{h}_{ij,\tau}^{\beta}, \bar{h}_{ij,c}^{\beta}\}, \max\{\bar{h}_{ij,\tau}^{\beta}, \bar{h}_{ij,c}^{\beta}\}]$.

In the automatic mechanism, the experts would not provide the new preference values. In this situation, we could adopt the following equation to generate the new value $rh_{ij,\tau}^{\beta}$:

$$rh_{ij,\tau}^{\beta} = \lambda \cdot \bar{h}_{ij,\tau}^{\beta} + (1 - \lambda) \cdot \bar{h}_{ij,c}^{\beta} \tag{35}$$

It is evident that if the new preference values are fulfilled, they will be closer to the group values than the old ones, and thus the consensus level improves.

Theorem 1. Let $H_\tau = (h_{ij,\tau})_{n \times n}$ (τ = 1, 2, ..., m) be the individual HFPRs, let CCL$^{(l)}$ be the consistency/consensus level which is generated by the above consensus reaching process. Then, we have CCL$^{(k+1)}$ > CCL$^{(k)}$ for each iteration k.

Proof. See Appendix A.

4.6. Selection process

When the group consensus is satisfied, we have to select the best alternatives according to the above collective NHFPR. Although we can use the existing hesitant aggregation operators [40] to obtain the overall preference degree $\bar{h}_{ic}$ of the alternative x_i, these operators must carry out the addition or multiplication operations on the combination of the elements of the input hesitant fuzzy arguments. Consequently, the number of the $\bar{h}_{ic}$ values will increase greatly, which will make the computation process very complicated and the result not accurately. In order to overcome these limitations, we apply the score function of a HFE to propose a Score HFPR (S-HFPR) to simplify the computation.

Definition 16. Let $H = (h_{ij})_{n \times n}$ be HFPR, then $SH = (sh_{ij})_{n \times n}$ is called the Score HFPR (SHFPR) of H, where $sh_{ij} = s(h_{ij}) = \frac{1}{\#h_{ij}} \sum_{\beta=1}^{\#h_{ij}} h_{ij}^{\beta}$.

Theorem 2. Let $H = (h_{ij})_{n \times n}$ be HFPR, $SH = (sh_{ij})_{n \times n}$ is the Score HFPR (SHFPR) associated to H, then SH is a FPR.

Proof. See Appendix B.

As SH is a FPR, here, we adopt a quantifier-guided choice degree of alternatives [4]: the *quantifier-guided dominance degree* (QGDD) [52] to select the best alternative, which is based on the OWA operator.

Definition 17. [51]. An OWA operator of dimension n is a mapping $OWA: R^n \rightarrow R$, which has an associated weighting vector $W = (w_1, w_2, ..., w_n)^T$ with $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, such that

$$\tag{36}$$

where b_j is the jth largest value in the set $\{a_1, a_2, ..., a_n\}$.

Yager [52] proposed the following expression to calculate the OWA weights:

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right) \tag{37}$$

where Q is a Basic Unit-interval Monotone (BUM) function.

Definition 18. (Hesitant Quantifier Guided Dominance Degree (HQGDD)). Let $X = \{x_1, x_2, ..., x_n\}$ be a set of alternatives, and $H = (h_{ij})_{n \times n}$ be HFPR which is provided by an expert against a particular criterion for the alternatives, $SH = (sh_{ij})_{n \times n}$ is the Score HFPR (SHFPR) of H, and Q a BUM function. The hesitant quantifier guided dominance degree associated to the alternative x_i, HQGDD$_i$ is defined as follows:

$$HQGDD_i = OWA_Q(sh_{ij}; j \neq i) \tag{38}$$

Where OWA_Q is an OWA operator whose weights are defined using a relative quantifier Q, and whose components are the elements of the corresponding row of SH, that is for x_i the set of $n - 1$ values $sh_{ij} \{sh_{ij}|j = 1, 2, ..., n$, and $i \neq j\}$.

The larger HQGDD$_i$, the better x_i is. Therefore, we can rank the alternatives x_i according to the value HQGDD$_i$, and select the best one.

Remark 7. Chiclana, et al. [7] defined QGDD and *quantifier-guided non-dominance degree* (QGNDD) to solve MCDM problems under FPRs. For simplicity, here, we only use QGDD to select the best alternative.

Remark 8. In the above, we have presented the consensus reaching process, which involves some steps and computations. In order to facilitate the decision making for the experts, a decision support system will be developed as an aid tool in a future study.

5. A case of study and comparative analysis

In this section, the consensus approach proposed here is applied for water allocation management to illustrate its real applicability and feasibility.

5.1. A case of study

This case of study is located in Jiangxi Province, China. The water resource of irrigation in the plain of Jiangxi Province has the function of flood protection, residential water in Jiangxi Basin. Because of the contradiction between the water demand and water allocation for different areas, it always emerges the water crisis in the Jiangxi Basin. Therefore, it needs efficient water allocation methods. Considering the characteristics of the multiple attributes for water resource systems, allocation methods should adopt the multiple attribute decision making method to select the most allocation alternative.

The water allocation alternative set has the following four alternatives: (1) The first alternative x_1 considers the social factor. The water allocation must be able to ensure the life stability of people. (2) The second alternative x_2 considers the economic factor. After the basic demands for different users, it focuses on the allocation to promote the local economic growth. (3) The third alternative x_3 thinks of the ecological factors. It increases efforts to protect the local ecological environment. (4) The fourth alternative x_4 considers the resource output and return of the local important scare resources. It only carries out the water allocation in the premise of ensuring the improvement and optimization of the local environment.

A committee that consists of four experts e_τ ($\tau = 1$–4) from different areas are organized to offer their assessments on the four alternatives x_i ($i = 1$–4). When the experts e_τ ($\tau = 1$–4) compare each pair of alternatives, they could not express their satisfaction exactly by one numeric value, and give several values. These assessments consist of the HFPRs H_τ ($\tau = 1$–4)

$$
H_1 = \begin{pmatrix}
\{0.5\} & \{0.3\} & \{0.5, 0.7\} & \{0.4\} \\
\{0.7\} & \{0.5\} & \{0.7, 0.9\} & \{0.8\} \\
\{0.5, 0.3\} & \{0.3, 0.1\} & \{0.5\} & \{0.6, 0.7\} \\
\{0.6\} & \{0.2\} & \{0.4, 0.3\} & \{0.5\}
\end{pmatrix},
$$

$$
H_2 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.5\} & \{0.1, 0.2\} & \{0.6\} \\
\{0.7, 0.5\} & \{0.5\} & \{0.7, 0.8\} & \{0.1, 0.3, 0.5\} \\
\{0.9, 0.8\} & \{0.3, 0.2\} & \{0.5\} & \{0.5, 0.6, 0.7\} \\
\{0.4\} & \{0.9, 0.7, 0.5\} & \{0.5, 0.4, 0.3\} & \{0.5\}
\end{pmatrix},
$$

$$
H_3 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.5\} & \{0.7\} & \{0.7, 0.8\} \\
\{0.7, 0.5\} & \{0.5\} & \{0.2, 0.3, 0.4\} & \{0.5, 0.6\} \\
\{0.3\} & \{0.8, 0.7, 0.6\} & \{0.5\} & \{0.7, 0.8, 0.9\} \\
\{0.3, 0.2\} & \{0.5, 0.4\} & \{0.3, 0.2, 0.1\} & \{0.5\}
\end{pmatrix},
$$

$$
H_4 = \begin{pmatrix}
\{0.5\} & \{0.4, 0.5, 0.6\} & \{0.3, 0.4\} & \{0.5, 0.7\} \\
\{0.6, 0.5, 0.4\} & \{0.5\} & \{0.3\} & \{0.6, 0.7, 0.8\} \\
\{0.7, 0.6\} & \{0.7\} & \{0.5\} & \{0.8, 0.9\} \\
\{0.5, 0.3\} & \{0.4, 0.3, 0.2\} & \{0.2, 0.1\} & \{0.5\}
\end{pmatrix}.
$$

Step 1. Normalization process: We use Definition 7 and the additive-consistency based estimation process to add the elements until all the comparison elements have the same number of values and obtain the NHFPRs. The possible $\bar{H}_\tau$ ($\tau = 1$–4) are in the following:

$$
\bar{H}_1 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.4, 0.3\} & \{0.5, 0.7, 0.7\} & \{0.4, 0.9, 0.6\} \\
\{0.7, 0.6, 0.7\} & \{0.5\} & \{0.7, 0.9, 0.9\} & \{0.8, 1, 0.8\} \\
\{0.5, 0.3, 0.3\} & \{0.3, 0.1, 0.1\} & \{0.5\} & \{0.6, 0.7, 0.4\} \\
\{0.6, 0.1, 0.4\} & \{0.2, 0, 0.2\} & \{0.4, 0.3, 0.6\} & \{0.5\}
\end{pmatrix},
$$

$$
\bar{H}_2 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.5, 0.6\} & \{0.1, 0.2, 0.4\} & \{0.6, 0.3, 0.6\} \\
\{0.7, 0.5, 0.4\} & \{0.5\} & \{0.7, 0.8, 0.3\} & \{0.1, 0.3, 0.5\} \\
\{0.9, 0.8, 0.6\} & \{0.3, 0.2, 0.7\} & \{0.5\} & \{0.5, 0.6, 0.7\} \\
\{0.4, 0.7, 0.4\} & \{0.9, 0.7, 0.5\} & \{0.5, 0.4, 0.3\} & \{0.5\}
\end{pmatrix},
$$

$$
\bar{H}_3 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.5, 0.8\} & \{0.7, 0.4, 0.7\} & \{0.7, 0.8, 1\} \\
\{0.7, 0.5, 0.2\} & \{0.5\} & \{0.2, 0.3, 0.4\} & \{0.5, 0.6, 0.8\} \\
\{0.3, 0.6, 0.3\} & \{0.8, 0.7, 0.6\} & \{0.5\} & \{0.7, 0.8, 0.9\} \\
\{0.3, 0.2, 0\} & \{0.5, 0.4, 0.2\} & \{0.3, 0.2, 0.1\} & \{0.5\}
\end{pmatrix},
$$

$$\bar{H}_4 = \begin{pmatrix} \{0.5\} & \{0.4, 0.5, 0.6\} & \{0.3, 0.4, 0.4\} & \{0.5, 0.7, 0.9\} \\ \{0.6, 0.5, 0.4\} & \{0.5\} & \{0.3, 0.35, 0.3\} & \{0.6, 0.7, 0.8\} \\ \{0.7, 0.6, 0.6\} & \{0.7, 0.65, 0.7\} & \{0.5\} & \{0.8, 0.9, 1\} \\ \{0.5, 0.3, 0.1\} & \{0.4, 0.3, 0.2\} & \{0.2, 0.1, 0\} & \{0.5\} \end{pmatrix}.$$

Step 2. Compute consistency levels: For each NHFPR $\bar{H}_\tau$ ($\tau = 1$–4), we obtain its associated additive consistency CH_τ by Eq. (13), and compute the corresponding consistency levels: pairs of alternatives cl_{ij}^τ, alternatives cl_i^τ, relation cl^τ, and the global consistency level CL.

Level 1. The consistency degrees for each pair of alternatives in each NHFPR $\bar{H}_\tau$ are:

$$CL^1 = \begin{pmatrix} 1 & 0.975 & 0.975 & 0.9667 \\ 0.975 & 1 & 0.9833 & 0.975 \\ 0.975 & 0.9833 & 1 & 0.975 \\ 0.9667 & 0.975 & 0.975 & 1 \end{pmatrix},$$

$$CL^2 = \begin{pmatrix} 1 & 0.925 & 0.875 & 0.9 \\ 0.925 & 1 & 0.8167 & 0.8417 \\ 0.8750 & 0.8167 & 1 & 0.9417 \\ 0.9 & 0.8417 & 0.9417 & 1 \end{pmatrix},$$

$$CL^3 = \begin{pmatrix} 1 & 0.8833 & 0.9250 & 0.9583 \\ 0.8833 & 1 & 0.9250 & 0.9583 \\ 0.9250 & 0.9250 & 1 & 0.9833 \\ 0.9583 & 0.9583 & 0.9833 & 1 \end{pmatrix},$$

$$CL^4 = \begin{pmatrix} 1 & 0.9875 & 0.9708 & 0.9833 \\ 0.9875 & 1 & 0.9917 & 0.9958 \\ 0.9708 & 0.9917 & 1 & 0.9792 \\ 0.9833 & 0.9958 & 0.9792 & 1 \end{pmatrix}.$$

Level 2. The alternatives consistency levels are:

$$(cl_i^1) = (0.9722, 0.9778, 0.9778, 0.9722),$$

$$(cl_i^2) = (0.9, 0.8611, 0.8778, 0.8944),$$

$$(cl_i^3) = (0.9222, 0.9222, 0.9444, 0.9667),$$

$$(cl_i^4) = (0.9806, 0.9917, 0.9806, 0.9861).$$

Level 3. The individual consistency levels are:

$$cl^1 = 0.975, \ cl^2 = 0.8833, \ cl^3 = 0.9389, \ cl^4 = 0.9847.$$

The global consistency degree is:

$$CL = \frac{0.975 + 0.8833 + 0.9389 + 0.9847}{4} = 0.9455$$

Step 3. Compute the consensus degree. For each pair of HFPRs $\bar{H}_\mu$ and $\bar{H}_\tau$, ($\mu < \tau$), we compute their similarity matrix $SM^{\mu\tau} = (sm_{ij}^{\mu\tau})$ by Eq. (19). Then, using Eq. (20) to obtain the collective similarity matrix SM, consensus levels on alternatives ca_i, consensus level on relation CR.

Level 1. The collective similarity matrix. In this case study, there are four experts, there are $4 \times 3/2 = 6$ similarity matrices $SM^{\mu\tau}$ ($\mu < \tau$), and the collective one by the arithmetic mean aggregation operator is:

$$SM = \begin{pmatrix} 1 & 0.9111 & 0.7556 & 0.7778 \\ 0.9111 & 1 & 0.6639 & 0.7056 \\ 0.7556 & 0.6639 & 1 & 0.7778 \\ 0.7778 & 0.7056 & 0.7778 & 1 \end{pmatrix}.$$

Level 2. Consensus levels on alternatives. Based on the similarity matrix SM, we compute the consensus levels by Eq. (22) on each alternative are:

$$ca_1 = 0.8148, \ ca_2 = 0.7602, \ ca_3 = 0.7324, \ ca_4 = 0.7537.$$

Level 3. Consensus on the relation CR by Eq. (23) is:

$$CR = 0.7653$$

Table 1
IOWA weights by different parameter a.

a	w_1	w_2	w_3	w_4	w_1-w_2	w_2-w_3	w_3-w_4
0.1	0.8741	0.0623	0.0374	0.0262	0.8118	0.0249	0.0111
0.2	0.7640	0.1127	0.0714	0.0518	0.6513	0.0413	0.0196
0.3	0.6679	0.1532	0.1023	0.0767	05147	0.0509	0.0256
0.4	0.5838	0.1850	0.1303	0.1009	0.3988	0.0547	0.0294
0.5	0.5103	0.2096	0.1556	0.1245	0.3007	0.0540	0.0311
0.6	0.4460	0.2280	0.1784	0.1475	0.2180	0.0496	0.0309
0.7	0.3899	0.2413	0.1989	0.1699	0.1486	0.0423	0.0291
0.8	0.3408	0.2502	0.2173	0.1917	0.0906	0.0329	0.0256
0.9	0.2979	0.2555	0.2337	0.2129	0.0424	0.0218	0.0208

Step 4. Proximity measure: The collective HFPR $\bar{H}_c$ is necessary to compute proximity measures for each expert. In order to use IOWA to aggregate the individual HFPRs into a collective one, one important step is to obtain the weights. Yager and Filev [53] proposed the following monotone function:

$$Q(x) = x^a, \ 0 < a < 1 \tag{39}$$

It is important to set parameter a in Eq. (39). Table 1 shows the weighting vector when setting different parameter a. Here, since $cl^1 = 0.975$, $cl^2 = 0.8833$, $cl^3 = 0.9389$, $cl^4 = 0.9847$, we have: $\sigma(1) = 4$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 2$. From Table 1, we can see that the smaller a, the smaller the weight corresponding to cl^2, but the large the value of $w_1 - w_2$ are. As the differences among cl^τ ($\tau = 1$–4) is not too large, therefore, we also make the differences of the given weights not too large, in this paper, we adopt $a = 0.9$ to compute and obtain

$$w_1 = 0.2979, \ w_2 = 0.2555, \ w_3 = 0.2337, \ w_4 = 0.2129.$$

By Eq. (26), we have

$$\bar{H}_c = \begin{pmatrix} \{0.5\} & \{0.3298, 0.4744, 0.5\} & \{0.4020, 0.4341, 0.4767\} & \{0.5425, 0.6893, 0.7361\} \\ \{0.6702, 0.5256, 0.5\} & \{0.5\} & \{0.4640, 0.5747, 0.4767\} & \{0.5213, 0.6681, 0.7361\} \\ \{0.5980, 0.5659, 0.5233\} & \{0.5360, 0.4253, 0.5233\} & \{0.5\} & \{0.6617, 0.7617, 0.7595\} \\ \{0.4575, 0.3107, 0.2639\} & \{0.4787, 0.3319, 0.2639\} & \{0.3383, 0.2383, 0.2405\} & \{0.5\} \end{pmatrix}.$$

As $\bar{H}_c$ is obtained, the computation of the proximity measures on different levels could be carried out:
Level 1. Proximity pp^τ on pairs of alternatives (x_i, x_j)

$$pp^1 = \begin{pmatrix} 1 & 0.8986 & 0.8042 & 0.8369 \\ 0.8986 & 1 & 0.6718 & 0.7752 \\ 0.8042 & 0.6718 & 1 & 0.8391 \\ 0.8369 & 0.7752 & 0.8391 & 1 \end{pmatrix},$$

$$pp^2 = \begin{pmatrix} 1 & 0.9482 & 0.7958 & 0.8057 \\ 0.9482 & 1 & 0.7873 & 0.6582 \\ 0.7958 & 0.7873 & 1 & 0.8724 \\ 0.8057 & 0.6582 & 0.8724 & 1 \end{pmatrix},$$

$$pp^3 = \begin{pmatrix} 1 & 0.9815 & 0.8638 & 0.8893 \\ 0.9815 & 1 & 0.7949 & 0.9489 \\ 0.8638 & 0.7949 & 1 & 0.9276 \\ 0.8893 & 0.9489 & 0.9276 & 1 \end{pmatrix},$$

$$pp^4 = \begin{pmatrix} 1 & 0.9347 & 0.9291 & 0.9276 \\ 0.9347 & 1 & 0.8116 & 0.9418 \\ 0.9291 & 0.8116 & 1 & 0.8276 \\ 0.9276 & 0.9418 & 0.8276 & 1 \end{pmatrix}.$$

Level 2. Proximity measure pa_i^τ on alternatives.

$$pa^1 = (0.8466, 0.7818, 0.7717, 0.8171),$$

$$pa^2 = (0.8499, 0.7979, 0.8185, 0.7788),$$

$$pa^3 = (0.9115, 0.9084, 0.8621, 0.9219),$$

$$pa^4 = (0.9305, 0.8960, 0.8561, 0.8990).$$

Level 3. Proximity measure pr^τ on relation.

$pr^1 = 0.8043, \ pr^2 = 0.8113, \ pr^3 = 0.9010, \ pr^3 = 0.9010, \ pr^4 = 0.8954.$

Step 5. Consistency/consensus level controlling stage. For illustration purpose, we assume $\lambda = 0.2$, which denotes that the consensus level is more important than the consistency level, and

$CCL = 0.2 \cdot 0.9455 + (1 - 0.2) \cdot 0.7653 = 0.8013.$

Setting the minimum CCL threshold value $\gamma = 0.85$. As $CCL < \gamma$, then feedback mechanism is activated.

Step 6. Feedback mechanism. The recommendation is offered to help the expert to update his/her opinions to improve the CCL.
(1) Identify the experts EXPCH:

$EXPCH = \{1,2\}$

e_1, e_2 should revise their preferences
(2) Identify the alternatives:

$ALT = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$

Identify the preference values:

$APS = \{(1, 2, 3), (1, 2, 4), (1, 3, 2), (1, 4, 2), (2, 2, 3), (2, 2, 4), (2, 3, 2), (2, 4, 2)\}$

Therefore, the recommendations are:
To expert $e_1 \Rightarrow$ You should provide values $r\bar{h}_{23}^1 \in [0.4640, 0.7], r\bar{h}_{23}^2 \in [0.5747, 0.9], \ r\bar{h}_{23}^3 \in [0.4767, 0.9].$
To expert $e_1 \Rightarrow$ You should provide values $r\bar{h}_{24}^1 \in [0.5213, 0.8], \ r\bar{h}_{24}^2 \in [0.6681, 1], \ r\bar{h}_{24}^3 \in [0.7361, 0.8].$
To expert $e_1 \Rightarrow$ You should provide values $r\bar{h}_{32}^1 \in [0.3, 0.5360], r\bar{h}_{32}^2 \in [0.1, 0.4253], \ r\bar{h}_{32}^3 \in [0.1, 0.5233].$
To expert $e_1 \Rightarrow$ You should provide values $r\bar{h}_{42}^1 \in [0.2, 0.4787], \ r\bar{h}_{42}^2 \in [0, 0.3319], \ r\bar{h}_{24}^3 \in [0.2, 0.2639].$
To expert $e_2 \Rightarrow$ You should provide values $r\bar{h}_{23}^1 \in [0.4640, 0.7], r\bar{h}_{23}^2 \in [0.5747, 0.8], \ r\bar{h}_{23}^3 \in [0.3, 0.4767].$
To expert $e_2 \Rightarrow$ You should provide values $r\bar{h}_{24}^1 \in [0.1, 0.5213], r\bar{h}_{24}^2 \in [0.3, 0.6681], \ r\bar{h}_{24}^3 \in [0.5, 0.7361].$
To expert $e_2 \Rightarrow$ You should provide values $r\bar{h}_{32}^1 \in [0.3, 0.5360], r\bar{h}_{32}^2 \in [0.2, 0.4253], \ r\bar{h}_{32}^3 \in [0.5233, 0.7].$
To expert $e_2 \Rightarrow$ You should provide values $r\bar{h}_{42}^1 \in [0.4787, 0.9], r\bar{h}_{42}^2 \in [0.3319, 0.7], \ r\bar{h}_{24}^3 \in [0.2639, 0.5].$
The experts e_1 and e_2 provide the following new preferences

$$
\bar{H}_1 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.4, 0.3\} & \{0.5, 0.7, 0.7\} & \{0.4, 0.9, 0.6\} \\
\{0.7, 0.6, 0.7\} & \{0.5\} & \{0.5, 0.6, 0.6\} & \{0.6, 0.8, 0.8\} \\
\{0.5, 0.3, 0.3\} & \{0.5, 0.4, 0.4\} & \{0.5\} & \{0.6, 0.7, 0.4\} \\
\{0.6, 0.1, 0.4\} & \{0.4, 0.2, 0.2\} & \{0.4, 0.3, 0.6\} & \{0.5\}
\end{pmatrix},
$$

$$
\bar{H}_2 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.5, 0.6\} & \{0.1, 0.2, 0.4\} & \{0.6, 0.3, 0.6\} \\
\{0.7, 0.5, 0.4\} & \{0.5\} & \{0.5, 0.6, 0.4\} & \{0.5, 0.6, 0.7\} \\
\{0.9, 0.8, 0.6\} & \{0.5, 0.4, 0.6\} & \{0.5\} & \{0.5, 0.6, 0.7\} \\
\{0.4, 0.7, 0.4\} & \{0.5, 0.4, 0.3\} & \{0.5, 0.4, 0.3\} & \{0.5\}
\end{pmatrix}.
$$

Applying the same process, CL, CR, and CCL are:

$CL = 0.9462, \ CR = 0.8282, \ CCL = 0.8518.$

If the experts will not give their new preferences, and adopt the automatic recommendation values, then the new preference relations for experts e_1 and e_2 are:

$$
\bar{H}_1 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.4, 0.3\} & \{0.5, 0.7, 0.7\} & \{0.4, 0.9, 0.6\} \\
\{0.7, 0.6, 0.7\} & \{0.5\} & \{0.5112, 0.6393, 0.5613\} & \{0.577, 0.7345, 0.7489\} \\
\{0.5, 0.3, 0.3\} & \{0.4888, 0.3603, 0.4387\} & \{0.5\} & \{0.6, 0.7, 0.4\} \\
\{0.6, 0.1, 0.4\} & \{0.423, 0.2655, 0.2511\} & \{0.4, 0.3, 0.6\} & \{0.5\}
\end{pmatrix}
$$

$$
\bar{H}_2 = \begin{pmatrix}
\{0.5\} & \{0.3, 0.5, 0.6\} & \{0.1, 0.2, 0.4\} & \{0.6, 0.3, 0.6\} \\
\{0.7, 0.5, 0.4\} & \{0.5\} & \{0.5112, 0.6197, 0.4413\} & \{0.4370, 0.5945, 0.6889\} \\
\{0.9, 0.8, 0.6\} & \{0.4888, 0.3803, 0.5587\} & \{0.5\} & \{0.5, 0.6, 0.7\} \\
\{0.4, 0.7, 0.4\} & \{0.5630, 0.4055, 0.3111\} & \{0.5, 0.4, 0.3\} & \{0.5\}
\end{pmatrix}
$$

Similarly, we have:

$CL = 0.9437, \ CR = 0.8266, \ CCL = 0.8500.$

In the interactive mechanism, the global CL increases a little, but in automatic mechanism, it decreases a little. In addition, both CRs increase, and CCLs are both satisfying the minimum consensus threshold 0.85 after one iteration modification. However, based on our

carefully calculation, in the interactive mechanism, if the revised values offered by the experts are closer to the collective values, the easier the CCL reaches to the threshold. Otherwise, it will need more iteration to reach its predefined threshold.

$$
\bar{H}_c = \begin{pmatrix}
\{0.5\} & \{0.3298, 0.4766, 0.5051\} & \{0.4013, 0.4260, 0.4702\} & \{0.5480, 0.6831, 0.7389\} \\
\{0.6702, 0.5234, 0.4949\} & \{0.5\} & \{0.3714, 0.4649, 0.4170\} & \{0.5339, 0.6601, 0.7635\} \\
\{0.5987, 0.5740, 0.5298\} & \{0.6286, 0.5350, 0.5830\} & \{0.5\} & \{0.6623, 0.7623, 0.7687\} \\
\{0.4520, 0.3169, 0.2611\} & \{0.4661, 0.3399, 0.2365\} & \{0.3377, 0.2377, 0.2313\} & \{0.5\}
\end{pmatrix}
$$

Step 7. Selection process. As $\bar{H}_c$ is obtained, we apply Definition 16 to get the SH_c as follows:

$$
SH_c = \begin{pmatrix}
0.5 & 0.4372 & 0.4325 & 0.6567 \\
0.5628 & 0.5 & 0.4178 & 0.6525 \\
0.5675 & 0.5822 & 0.5 & 0.7311 \\
0.3433 & 0.3475 & 0.2689 & 0.5
\end{pmatrix}
$$

Using the BUM function $Q = x^{0.9}$, the weighting vector of OWA operator is: $W = (0.3720, 0.3222, 0.3057)^T$, and then the $HQGDD$ are: $HQGDD_1 = 0.5174$, $HQGDD_2 = 0.5518$, $HQGDD_3 = 0.6330$, $HQGDD_4 = 0.3221$.
According to the degrees $QGDD_i$, the ranking of alternatives are as follows:

$$
x_3 \succ x_2 \succ x_1 \succ x_4
$$

Thus, the ranking alternatives are: ecological factor, economic factor, social factor and output and return factor. The best alternative is x_3 (Ecological factor).

The results show that the ecological factor is the most important factor. The government should first protect the ecosystem to ensure a good environment. The second best alternative is the economic factor. As Jiangxi Province is a relatively backward area in China, it needs to develop its local economic to guarantee continued investment to protect flood and other problems. The third best alternative is social factor. While developing the economic, the local government should consider the life stability of people, and finally, the output and return is the least factor at present.

5.2. Comparative analysis

In this section, we compare our approach with the existing methods.

Zhang et al. [55] developed a consensus decision support model for GDM with HFPRs. For the above example, we use Zhang et al. [55]'s method to obtain the best alternative includes the following steps.

First, Zhang et al. [55] proposed a normalization method and used the following

$$
\tilde{h}_{ij,\tau}^{(l)} = \frac{1}{n} \overset{n}{\underset{t=1}{\oplus}} \left(h_{it,\tau}^{(l)} \oplus h_{tj,\tau}^{(l)} \right) \ominus \frac{1}{2} \tag{40}
$$

$$
CI = (H_\tau^{(l)}) = D \left(H_\tau^{(l)}, \tilde{H}_\tau^{(l)} \right) \tag{41}
$$

to compute its consistent HFPR and consistency index. If the consistency index of some of the HFPR is smaller than the predefined threshold, Zhang et al. [55] used the following:

$$
h_{ij,\tau}^{(l+1)} = \begin{cases} \delta h_{ij,\tau}^{(l)} \oplus (1-\delta)\tilde{h}_{ij,\tau}^{(l)} \\ h_{ij,\tau}^{(l)} \end{cases} \tag{42}
$$

to obtain a modified NHFPR.

For the above example, Zhang et al. [55] first got normalized HFPRs and got their consistency index are: $cl^1 = 0.9306$, $cl^2 = 0.8313$, $cl^3 = 0.8944$ and $cl^4 = 0.9694$.

As the consistency indices are smaller than the predefined threshold, Zhang, et al. [55] obtained the following consistent HFPRs after two iterations:

$$
H_1^{(2)} = \begin{pmatrix}
\{0.5\} & \{0.255, 0.255, 0.255\} & \{0.455, 0.455, 0.5875\} & \{0.49, 0.49, 0.5575\} \\
\{0.745, 0.745, 0.745\} & \{0.5\} & \{0.7, 0.7, 0.8325\} & \{0.755, 0.755, 0.8225\} \\
\{0.545, 0.545, 0.4125\} & \{0.3, 0.3, 0.1675\} & \{0.5\} & \{0.555, 0.555, 0.52\} \\
\{0.51, 0.51, 0.4425\} & \{0.245, 0.245, 0.1775\} & \{0.445, 0.445, 0.48\} & \{0.5\}
\end{pmatrix};
$$

$$
H_2^{(2)} = \begin{pmatrix}
\{0.5\} & \{0.3742, 0.35, 0.3763\} & \{0.3228, 0.3292, 0.398\} & \{0.303, 0.3958, 0.5257\} \\
\{0.6258, 0.65, 0.6238\} & \{0.5\} & \{0.4525, 0.4837, 0.5278\} & \{0.4218, 0.5413, 0.6485\} \\
\{0.6772, 0.6708, 0.602\} & \{0.3, 0.3, 0.1675\} & \{0.5\} & \{0.4753, 0.5629, 0.6258\} \\
\{0.697, 0.6042, 0.4743\} & \{0.5782, 0.4587, 0.3515\} & \{0.5247, 0.4371, 0.3743\} & \{0.5\}
\end{pmatrix};
$$

$$H_3^{(2)} = \begin{pmatrix} \{0.5\} & \{0.5722,0.6485,0.6237\} & \{0.4772, 0.5515, 0.5515\} & \{0.6505, 0.8000, 0.8247\} \\ \{0.4278,0.3515,0.3763\} & \{0.5\} & \{0.3980, 0.3990, 0.4248\} & \{0.5742, 0.6495, 0.6990\} \\ \{0.5228, 0.4485, 0.4485\} & \{0.602, 0.601, 0.5752\} & \{0.5\} & \{0.6753, 0.7505, 0.7762\} \\ \{0.3495, 0.2, 0.1753\} & \{0.4258, 0.3505, 0.301\} & \{0.3248, 0.2495, 0.2238\} & \{0.5\} \end{pmatrix};$$

$$H_4^{(2)} = \begin{pmatrix} \{0.5\} & \{0.4225, 0.5, 0.555\} & \{0.255, 0.3239, 0.3775\} & \{0.5225, 0.6709, 0.7675\} \\ \{0.5775, 0.5, 0.445\} & \{0.5\} & \{0.3225, 0.3225, 0.3225\} & \{0.6, 0.6775, 0.7325\} \\ \{0.745, 0.6708, 0.6225\} & \{0.6775, 0.6775, 0.6775\} & \{0.5\} & \{0.7775, 0.8517, 0.9\} \\ \{0.4775, 0.3291, 0.2325\} & \{0.4, 0.3225, 0.2675\} & \{0.2225, 0.1483, 0.1\} & \{0.5\} \end{pmatrix}.$$

In the consensus reaching process, if the group consensus index $GCI(H_\tau^{(l)})$ $(\tau = 1, 2, ..., m)$, where $GCI(H_\tau^{(l)}) = d\left(H_\tau^{(l)}, H_c^{(l)}\right)$ is smaller than the predefined threshold, Zhang et al. [55] utilized the following:

$$h_{ij,\tau}^{(l+1)} = \eta h_{ij,\tau}^{(l)} \oplus (1 - \eta) h_{ij,c}^{(l)} \tag{43}$$

to update the HFPRs.

Compared with Zhang et al. [55]'s method, the proposed method has the following advantages:

(1) The existing definitions [42,55] of HFPR need to reorder the values in each HFE, which not only distorts the experts' primitive information, but also destroys the additive consistency. In the above example, Zhang, et al. [55] obtained the acceptable consistent $H_2^{(2)}$ by Eq. (42). The HFE $h_{23,2}^{(2)} = \{0.3742, 0.35, 0.3763\}$, $h_{23,3}^{(2)} = \{0.5722, 0.6485, 0.6273\}$, the values in $h_{23,2}^{(2)}$, $h_{23,3}^{(2)}$ are not arranged in ascending order, which do not conform to their definition of HFRP (i.e., $h_{ij}^{\sigma(\beta)} < h_{ij}^{\sigma(\beta+1)}$ does not hold). If the values in $h_{23,2}^{(2)}$, $h_{23,3}^{(2)}$ are rearranged in increasing order, $H_2^{(2)}$, $H_3^{(2)}$ are not additively consistent HFPRs according to their definition. Therefore, the existing definition is not appropriate. In this paper, we redefine the concept of the HFPR, which does not arrange the values in the hesitant element in ascending or descending order. This aims to not only maintain the experts' original information as much as possible, but preserves the additive consistency property.

(2) In the normalization process, we propose a definition to add the elements, and also a modified procedure to estimate the unknown values. We look the values as missing values in the short number of hesitant elements, and then propose a revised additive consistency-based estimation procedure to add the unknown values, which also can surmise the values as much as possible. Additionally, they can be used to deal with not only the complete HFPRs, but also the incomplete HFPRs, while Zhu, et al. [60], Zhang, et al. [55]'s methods add the elements randomly, and could not handle the incomplete HFPRs. Furthermore, in the estimation procedure, we take the reciprocity property into account, which is more simplicity and has less computation burden than the methods in [21,23]. Furthermore, the additive consistency property is used, and eventually, the normalized HFPR will be more consistent than the existing methods. In Example 3, the initial consistency levels for the four HFPRs are $cl^1 = 0.975$, $cl^2 = 0.8833$, $cl^3 = 0.9389$ and $cl^4 = 0.9847$, respectively. While they are $cl^1 = 0.9306$, $cl^2 = 0.8313$, $cl^3 = 0.8944$ and $cl^4 = 0.9694$, respectively, in Zhang et al. [55]'s method.

(3) In the feedback process, we propose two mechanisms: interactive mechanism and an automatic mechanism. In the interactive mechanism, we pay attention to the interventions of the experts, which can respect the experts' opinion. If the experts are unwilling to provide their new preferences, the automatic mechanism can be used to implement the consensus process. While Zhang et al. [55]'s method only takes the automatic mechanism. In addition, in the consensus process, we take the consistency and consensus degrees into consideration at the same time. While the method in Zhang et al. [55] takes the consistency and consensus degrees separately. The drawback of Zhang et al. [55]'s method is that the consistency degrees are changed in the consensus reaching process. Thus, it needs to check the consistency degree in the consensus reaching process, while our method considers the changes at the same time. Furthermore, in the feedback stage, only those values whose consistency/consensus levels are below the predefined threshold are modified. The rational is to retain the experts' original preference information. On the other hand, Zhang et al. [55] employed Eq. (42) to modify all the preference values in the consistency improving process, and Eq. (43) to modify all the preference values in the consensus reaching process by setting a parameter δ. Although their method can achieve the consistency and consensus levels respectively, the implication is that the final modified HFPRs are often significantly different from the original judgments expressed by the experts, which will make the results unreliable.

(4) It investigates the AC-IHFOWA operator to aggregate HFPRs taking into account the experts' consistency levels. The more consistent level, the larger the weight of the expert is. However, Zhang et al. [55] assigned the weights to the experts at the beginning, which do not change in the whole process, and there is no rule how to assign the weights. The experts with higher consistent level may be assigned with lower weights, which is not reasonable.

(5) Finally, in the aggregation phase, we aggregate the normalized HFPRs into a collective one via the extension principle, i.e., the collective value is the weighted aggregation value of the corresponding hesitant fuzzy values. Therefore, each of the collective hesitant fuzzy elements has the same number of values as the individual one. While using the existing hesitant fuzzy weighted averaging (HFWA) operator, hesitant fuzzy ordered weighted averaging (HFOWA) [40], it will has thousands of values in each of the aggregated hesitant fuzzy element for the GDM problem of this paper. Because the number of values in each aggregated hesitant element is the combination of all the aggregation arguments, it will lose accurate and even will distort the decision information, and eventually makes the results unreliable.

6. Conclusions

A consensus model, which is based on consensus and consistency degrees, for GDM problems with HFPRs has been developed in this paper. To do so, a revised definition of HFPRs and a new method to add the elements into a hesitant fuzzy set have been presented. In order to normalize the HFPRs, a revised additive consistency-based estimation procedure has been developed to estimate the unknown values, which are added to the short hesitant fuzzy sets. As it has been shown, the normalized HFPR is more consistent than the ones proposed by the existing methods.

Furthermore, two feedback mechanisms are incorporated to the consensus method: an interactive mechanism advising the experts how to give their new preference values, and an automatic mechanism carrying out the consensus reaching process without the experts' intervention. To do so, the AC-IHFOWA operator has been developed to aggregate the individual HFPRs into a collective one, which gives more importance to those HFPRs with higher consistency. Finally, a case of study for water allocation management in Jiangxi Province of China has been provided to illustrate the application and to show the advantages of the proposed model.

Acknowledgments

Appendix A. Proof of Theorem 1.

Based on Eqs. (14)-(23) and (28)-(31), we have

$$CCL^{(k+1)} = \lambda CL^{(k+1)} + (1-\lambda)CR^{(k+1)}$$

$$= \lambda \frac{\sum_{\tau=1}^{m} cl^{\tau,(k+1)}}{m} + (1-\lambda)\frac{\sum_{i=1}^{n} ca_i^{(k+1)}}{n}$$

$$= \lambda \frac{\sum_{\tau=1}^{m}\sum_{i=1}^{n} cl_i^{\tau,(k+1)}}{mn} + (1-\lambda)\frac{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} cp_{ij}^{(k+1)}}{n}$$

$$= \lambda \frac{\sum_{\tau=1}^{m}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(1-\varepsilon h_{ij}^{\tau,(k+1)})}{mn(n-1)} + (1-\lambda)\frac{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} sm_{ij}^{(k+1)}}{n}$$

As $\varepsilon h_{ij}^{\tau,(k+1)} < \varepsilon h_{ij}^{\tau,(k)}$, and $sm_{ij}^{(k+1)} > sm_{ij}^{(k)}$, then

$$CCL^{(k+1)} = \lambda \frac{\sum_{\tau=1}^{m}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(1-\varepsilon h_{ij}^{\tau,(k+1)})}{mn(n-1)} + (1-\lambda)\frac{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} sm_{ij}^{(k+1)}}{n}$$

$$> \lambda \frac{\sum_{\tau=1}^{m}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(1-\varepsilon h_{ij}^{\tau,(k)})}{mn(n-1)} + (1-\lambda)\frac{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} sm_{ij}^{(k)}}{n}$$

$$= \lambda \frac{\sum_{\tau=1}^{m}\sum_{i=1}^{n} cl_i^{\tau,(k)}}{mn} + (1-\lambda)\frac{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} cp_{ij}^{(k)}}{n}$$

$$= \lambda \frac{\sum_{\tau=1}^{m} cl^{\tau,(k)}}{m} + (1-\lambda)\frac{\sum_{i=1}^{n} ca_i^{(k)}}{n}$$

$$= \lambda CL^{(k)} + (1-\lambda)CR^{(k)} = CCL^{(k)}$$

That is:

$$CCL^{(k+1)} > CCL^{(k)}$$

Which completes the proof of Theorem 1.

Appendix B. Proof of Theorem 2.

Proof. Since $H = (h_{ij})_{n\times n}$ be HFPR, by Definition 5, we have that

$$h_{ij}^{\beta} + h_{ji}^{\beta} = 1$$

Therefore,

$$sh_{ij} + sh_{ji} = s(h_{ij}) + s(h_{ji}) = \frac{1}{\#h_{ij}}\sum_{\beta=1}^{\#h_{ij}} h_{ij}^{\beta} + \frac{1}{\#h_{ji}}\sum_{\beta=1}^{\#h_{ji}} h_{ij}^{\beta}$$

$$= \frac{1}{\#h_{ij}}\sum_{\beta=1}^{\#h_{ij}}(h_{ij}^{\beta} + h_{ji}^{\beta}) = 1$$

Which completes the proof of Theorem 2.

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