A rational and consensual method for group decision making with interval-valued intuitionistic multiplicative preference relations☆

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ABSTRACT

Interval-valued intuitionistic multiplicative variables (IVIMVs) can conveniently and effectively represent the uncertain multiplicative preferred and non-preferred judgements of decision makers, this paper studies group decision making (GDM) with interval-valued intuitionistic multiplicative preference relations (IVIMPRs). To calculate the interval-valued intuitionistic multiplicative priority weight vector reasonably, a new consistency concept is introduced that satisfies robustness and upper triangular property. Following this concept, models to judge the consistency and obtain consistent IVIMPRs from inconsistent ones are constructed, respectively. To address the problem of incomplete preferences, consistency-based model to determine missing values is built. Furthermore, we study the consensus for GDM with IVIMPRs and provide a new consensus approach. When an acceptable consensus level is not achieved, an interactive and automatic adjustment method is applied to reach a better consensus level. Following discussion about consistency and consensus, an algorithm for GDM is offered that can address inconsistent and incomplete IVIMPRs. Finally, a practical problem about selecting the steel supplier is selected to show the application of the new method.

1. Introduction

Because Zadeh's fuzzy sets can only express the preferences of the decision makers (DMs), Atanassov (1986) introduced intuitionistic fuzzy sets (IFSs) defined on [0, 1] to denote the preferred and non-preferred judgements of the DMs. Following Atanassov's work, many researchers studied intuitionistic fuzzy decision making. Taking the advantages of preference relations and intuitionistic fuzzy variables (IFVs), Szmidt and Kacprzyk (1988) introduced intuitionistic fuzzy preference relations (IFPRs). Considering the consistency of IFPRs, decision making with IFPRs based on additive consistency (AC) is studied in Chu et al. (2016) and Gong et al. (2011), and decision making with IFPRs based on multiplicative consistency (MC) is researched in Meng et al. (2017a) and Wu and Chiclana (2014).

Later, Xia et al. (2013) presented intuitionistic multiplicative sets (IMSs) that apply real values defined on Saaty's [1/9, 9] scale to denote the unbalanced preferred and non-preferred information. Using intuitionistic multiplicative variables (IMVs), Xia et al. (2013) introduced intuitionistic multiplicative preference relations (IMPRs) and defined two generalized intuitionistic multiplicative aggregation operators to rank objects. Considering the limitation of the score function on IMVs,

Garg (2017a) introduced an improved score function to increase the ranking among IMVs. Considering the interactive characteristics, Xia and Xu (2013) introduced the extended intuitionistic multiplicative Choquet ordered averaging (EIMCOA) operator. Garg (2017b) presented several distance measures between IMPRs and then offered a GDM method with IMPRs. Garg (2016, 2018b) presented some operations on IFVs, by which several intuitionistic fuzzy multiplicative interactive aggregation operators are defined. In addition, Garg (2016, 2018b) studied GDM with IMPRs based on the defined aggregation operators. Meanwhile, Garg (2018a) developed a correlation coefficient based GDM method with IMPRs. Jiang and Xu (2014) discussed group decision making (GDM) with IMPRs based on the consensus analysis. Jiang et al. (2013) discussed the MC of IMPRs similar to Saaty's concept, and Jiang et al. (2015) offered a convex combination consistency concept for IMPRs inspired by Liu (2009). Meng et al. (2019) noted and analyzed the limitations of Jiang et al.'s consistency concept and defined a new one. Then, the authors presented a new GDM with incomplete and inconsistent IMPRs. The application of decision making with intuitionistic multiplicative information can be seen in the literature (Luo et al., 2019; Liao et al., 2019).

However, IFPRs and IMPRs only permit the DMs to use real numbers to express the preferred and non-preferred recognitions, which may be still difficult. To address this issue, Xu (2007a) presented interval-valued intuitionistic fuzzy preference relations (IVIFPRs) whose elements are interval-valued intuitionistic fuzzy values (IVIFVs). Different from IFVs and IMVs, IVIFVs apply an interval in [0, 1] to express the uncertain preferred and non-preferred judgments of the DMs, respectively. Xu and Cai (2009) discussed the AC of IVIFPRs using the relationship between IVIFVs. Furthermore, they studied the MC of IVIFPRs using the multiplication laws defined on IVIFVs (Xu, 2007b). Following Tanino's MC (Tanino, 1984), Liao et al. (2014), and Xu and Cai (2009) separately offered another MC concept (MCC) for IVIFPRs. Wan et al. (2016) extended MCC for IMPRs and provided a new MCC for IVIFPRs. Wu and Chiclana (2012) presented intuitionistic quantifier guided non-dominance degree and gave a method to rank objects from IVIFPRs. GDM methods with IVIFPRs can be seen in the literature (Meng et al., 2018; Tang et al., 2018; Wan et al., 2018a,b).

To express the unbalanced uncertain preferred and non-preferred judgements of the DMs, Jiang and Xu (2014) generalized the notion of IMSs and presented interval-valued intuitionistic multiplicative sets (IVIMSs), whose elements are interval-valued intuitionistic multiplicative variables (IVIMVs). Considering the application of IVIMVs, the authors introduced several basic operational laws and then researched decision making with IVIMVs using normalized Minkowski and Manhattan distances. Recently, Zhang (2017) introduced IVIMVs for preference relations and presented interval-valued intuitionistic multiplicative preference relations (IVIMPRs). After that, the author provided two approaches to GDM with IVIMPRs using the aggregation operators. Based on some new operational laws on IVIMVs, Liu et al. (2019) researched their properties and presented an aggregation operator based GDM with IVIMPRs. Following the works of Liu (2009) and Jiang et al. (2015), Sahu et al. (2018) presented a consistency concept for IVIMPRs and offered a method for ascertaining missing values in incomplete IVIMPRs. Based on the acceptable consistency analysis, the authors offered a GDM with incomplete IVIMPRs. Based on Zhang and Prdrycz's consistency concept for IMPRs (Zhang and Prdrycz, 2017, 2019) offered a consistency concept for IVIMPRs. Using this consistency index for IM-PRs, the authors discussed the consistency of IVIFMPRs. Furthermore, Zhang and Prdrycz (2019) built several models for determining the weights of the DMs and calculating the interval-valued intuitionistic multiplicative priority weight vector (IVIMPWV). Following the work of Zhang and Prdrycz (2019) and Zhang and Prdrycz (2019) further discussed the consensus for GDM with IVIMPRs.

From the above literature review, one can check that studies about decision making with IVIMPRs are relatively fewer and there are some drawbacks.

(1) Most of previous studies (Jiang et al., 2014; Zhang, 2017; Liu et al., 2019) about GDM with IVIMPRs are based on the aggregation operators which disregard the consistency and/or consensus analysis. It means that such methods neither can ensure the rationality of ranking nor make the ranking represent the opinions of DMs.

(2) As some researchers noted the limitation of Liu's consistency concept (Liu, 2009), Sahu et al.'s consistency concept for IVIM-PRs (Sahu et al., 2018) has several drawbacks including (i) Sahu et al.'s consistency concept depends on the comparison order of objects, this concept is meaningless (Krejčí, 2017); (ii) when a given IVIMPR is inconsistent, different rankings may be obtained with respect to different comparison orders; (iii) different values for missing judgments may be obtained for different objects' compared orders. Meanwhile, different rankings may be derived too for incomplete IVIMPRs. Furthermore, Sahu et al.'s method (Sahu et al., 2018) for GDM with IVIMPRs does not make the consensus analysis and disregards how to determine the weights of the DMs. Moreover, this method can only address the situation where each object is compared at least once.

(3) As for methods in the literature (Zhang and Prdrycz, 2019; Zhang et al., 2019), the main drawbacks comes from the adopted consistency concept which is in fact based on the score function on IMVs (Xia et al., 2013): (i) when an IVIMPR is inconsistent, we cannot judge that it is caused by the lower/upper bound of interval preferred/non-preferred degree; (ii) following this concept, there may be infinite values for missing judgments. Furthermore, when an IVIMPR is completely consistent following Zhang and Prdrycz's consistency concept, there may be no IVIMPWV. Moreover, Zhang and Prdrycz's method disregarded the consistency analysis. Meanwhile, neither of them studied incomplete IVIMPRs.

To overcome the above mentioned drawbacks of previous research about GDM with IVIMPRs, this paper introduces a new GDM approach with IVIMPRs based on the consistency and consensus criteria to achieve a final ranking of objects/alternatives. In conclusion, the merits of the new method contain:

(i) The consistency of IVIMPRs is further analyzed, and a new MCC is offered that avoids the limitations of previous ones (Sahu et al., 2018; Zhang and Prdrycz, 2019).

(ii) Based on the new consistency concept, models to access the consistency of IVIMPRs from inconsistent ones are built that can achieve the minimum total adjustment and permitting to only adjust part judgments.

(iii) When incomplete IVIMPRs are obtained, a model to obtain unknown values is established that can address the case where ignored objects exist, that is, all their information is unknown. While all previous methods cannot cope with this case (Jiang and Xu, 2014; Zhang, 2017; Liu et al., 2019; Sahu et al., 2018; Zhang and Prdrycz, 2019; Zhang et al., 2019).

(iv) The consensus for GDM with IVIMPRs is studied. A consensus measure is defined, and a new method to improve the consensus level is constructed that can guarantee the consistency unchanged.

(v) A new distance measure based model is introduced to determine the DMS' weights. Finally, we show an application example of our new method.

For the aspect (ii), although methods in Sahu et al. (2018) and Zhang and Prdrycz (2019) also considered it, their rationality is questionable because of the drawbacks of the adopted consistency concepts. For aspects (iv) and (v), only Zhang et al.'s method (Zhang et al., 2019) studied them. However, the used consistency concept and the procedure for calculating the IVIMPWV have limitations as listed in (3).

The rest is organized as follows: Section 2 recalls basic concepts and two consistency concepts for IVIMPRs. Section 3 researches the consistency of IVIMPRs and offers a MCC for IVIMPRs. Section 4 studies incomplete and inconsistent IVIMPRs. Section 5 studies GDM with IVIMPRs. Section 6 uses a practical example to show the application. Conclusion is given in the end.

2. Preliminaries

In this section, there are two subsections. The first subsection introduces some basic concepts to help the readers understand the following discussion. The second subsection reviews two previous consistency concepts for IVIMPRs and analyzes their limitations.

2.1. Basic concepts

To denote the uncertain judgements, Saaty and Vargas (1987) utilized intervals defined on Saaty's [1/9, 9] scale to introduce interval-valued multiplicative preference relations (IVMPRs). First, we review the concept of intervals.

Definition 1. Let $\mathfrak{R}$ be the set of all real numbers. $\bar{a} = [a^-, a^+]$ is said to be an interval if $a^- \leq a^+$ with $a^-, a^+ \in \mathfrak{R}$, and $\bar{a}$ is said to be a positive interval if $a^- \leq a^+$ with $a^- > 0$.

Definition 2. Let $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ be any two positive intervals. Then, several of their operations are defined as follows:

(i) $\overline{a} \oplus \overline{b} = [a^- + b^-, a^+ + b^+]$;

(ii) $\overline{a} \otimes \overline{b} = [a^- b^-, a^+ b^+]$;

(iii) $\overline{a}/\overline{b} = [a^-/b^+, a^+/b^-]$;

(iv) $\overline{a}^\lambda = [(a^-)^\lambda, (a^+)^\lambda] \ \lambda \geq 0$;

(v) $\log_\lambda \overline{a} = [\log_\lambda(a^-), \log_\lambda(a^+)] \ 0 < \lambda \wedge \lambda \neq 1$.

Definition 3 (*Saaty and Vargas, 1987*). An IVMPR $\overline{B} = (\overline{b}_{ij})_{n \times n}$ on the object set $X = \{x_1, x_2, \ldots, x_n\}$ is defined as:

$$\overline{B} = \begin{pmatrix} [1,1] & [b_{12}^-, b_{12}^+] & \cdots & [b_{1n}^-, b_{1n}^+] \\ [b_{21}^-, b_{21}^+] & [1,1] & \cdots & [b_{2n}^-, b_{2n}^+] \\ \vdots & \vdots & \vdots & \vdots \\ [b_{n1}^-, b_{n1}^+] & [b_{n2}^-, b_{n2}^+] & \cdots & [1,1] \end{pmatrix} \quad (1)$$

where $b_{ij}^-, b_{ij}^+ > 0$, $b_{ij}^- \leq b_{ij}^+$ and $\begin{cases} b_{ij}^- = 1/b_{ji}^+ \\ b_{ij}^+ = 1/b_{ji}^-, \end{cases}$ $\overline{b}_{ij}$ shows that x_i is between b_{ij}^- and b_{ij}^+ times as important as x_j. When $b_{ij}^- = b_{ij}^+$ for all $i, j = 1, 2, \ldots, n$, $\overline{B}$ reduces to a multiplicative preference relation (*Saaty, 1980*).

After reviewing the previous consistency concepts for IVMPRs, Meng and Tan (2017) introduced the following MCC for IMPRs:

Definition 4 (*Meng and Tan, 2017*). Let $\overline{B} = (\overline{b}_{ij})_{n \times n}$ be an IVMPR. $\overline{B}$ is consistent if there is a consistent QIVMPR $\overline{Q} = (\overline{q}_{ij})_{n \times n}$, namely,

$$\overline{q}_{ij} = \overline{q}_{ik} \otimes \overline{q}_{kj} \quad (2)$$

for all $i, k, j = 1, 2, \ldots n$, where $\begin{cases} \overline{q}_{ij} = (\overline{b}_{ij})^{\lambda_{ij}} \otimes (\overline{b}_{ij}^\circ)^{1-\lambda_{ij}} \\ \overline{q}_{ji} = (\overline{b}_{ji}^\circ)^{\lambda_{ij}} \otimes (\overline{b}_{ji})^{1-\lambda_{ij}}, \end{cases}$ $i, j = 1, 2, \ldots n$, λ_{ij} is a 0–1 indicator variable (0-1-IV) such that $\lambda_{ij} = \begin{cases} 1 & \overline{q}_{ij} = \overline{b}_{ij} \wedge \overline{q}_{ji} = \overline{b}_{ji}^\circ \\ 0 & \overline{q}_{ij} = \overline{b}_{ij}^\circ \wedge \overline{q}_{ji} = \overline{b}_{ji}, \end{cases}$ and $\overline{b}_{ij}^\circ = [b_{ij}^+, b_{ij}^-]$ is the quasi interval for $\overline{b}_{ij}$, $i, j = 1, 2, \ldots n$.

For more explanations as well as the principle of Definition 4, please see Ref. Meng and Tan (2017). To denote the unbalanced uncertain positive and negative comparisons of one object over another, Jiang and Xu (2014) introduced the concept of IVIMSs.

Definition 5 (*Jiang and Xu, 2014*). An IVIMS $\tilde{D}$ on the object set $X = \{x_1, x_2, \ldots, x_n\}$ is defined as:

$$\tilde{D} = \{< x, \overline{\rho}_D(x), \overline{\sigma}_D(x) > | x \in X\} \quad (3)$$

which assigns to each element x an interval preferred degree $\overline{\rho}_D(x) = [\rho_D^-(x), \rho_D^+(x)]$ and an interval non-preferred degree $\overline{\sigma}_D(x) = [\sigma_D^-(x), \sigma_D^+(x)]$ with the conditions $\begin{cases} 1/a \leq \sigma_D^-(x), \sigma_D^+(x) \leq a \\ 1/a \leq \rho_D^-(x), \rho_D^+(x) \leq a \\ \rho_D^+(x)\sigma_D^+(x) \leq 1, \end{cases}$ where $[1/9, 9]$ is the given scale.

For convenience, $\tilde{\alpha} = (\overline{\rho}, \overline{\sigma})$ is called an IVIMV, where $\overline{\rho} = [\rho^-, \rho^+]$ and $\overline{\sigma} = [\sigma^-, \sigma^+]$ such that $\begin{cases} 1/9 \leq \sigma^-, \sigma^+ \leq 9 \\ 1/9 \leq \rho^-, \rho^+ \leq 9 \\ \rho^+ \sigma^+ \leq 1. \end{cases}$ Definition 5 shows when $\begin{cases} \rho_D^-(x) = \rho_D^+(x) \\ \sigma_D^-(x) = \sigma_D^+(x), \end{cases}$ then the IVIMS $\tilde{D}$ reduces to an IFMS. Furthermore, when $\begin{cases} \sigma^- = \sigma^+ \\ \rho^- = \rho^+, \end{cases}$ then the IVIMV $\tilde{\alpha}$ becomes the IFMN $\alpha = (\rho, \sigma)$.

Considering the order relationship between IVIMVs, the score and accuracy functions are defined as follows:

Definition 6 (*Jiang and Xu, 2014*). Let $\tilde{\alpha} = ([\rho^-, \rho^+], [\sigma^-, \sigma^+])$ be an IVIMV as defined in Definition 5. The score function is $S(\tilde{\alpha}) = \sqrt{\frac{\rho^- \rho^+}{\sigma^- \sigma^+}}$, and the accuracy function is $A(\tilde{\alpha}) = \sqrt{(\rho^+ \rho^-)(\sigma^+ \sigma^-)}$.

Let $\tilde{\alpha}_1 = ([\rho_1^-, \rho_1^+], [\sigma_1^-, \sigma_1^+])$ and $\tilde{\alpha}_2 = ([\rho_2^-, \rho_2^+], [\sigma_2^-, \sigma_2^+])$ be any two IVIMVs. Then, their order relationship is (*Jiang and Xu, 2014*):

$$\begin{cases} \tilde{\alpha}_1 > \tilde{\alpha}_2 & \text{if } S(\tilde{\alpha}_1) > S(\tilde{\alpha}_2) \\ \tilde{\alpha}_1 > \tilde{\alpha}_2 & \text{if } S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) \wedge A(\tilde{\alpha}_1) > A(\tilde{\alpha}_2) \\ \tilde{\alpha}_1 = \tilde{\alpha}_2 & \text{if } S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) \wedge A(\tilde{\alpha}_1) = A(\tilde{\alpha}_2) \end{cases} \quad (4)$$

Using IVIMVs, Zhang (2017) introduced IVIMPRs:

Definition 7 (*Zhang, 2017*). Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be defined on the object set $X = \{x_1, x_2, \ldots, x_n\}$, where $\tilde{r}_{ij} = (\overline{\rho}_{ij}, \overline{\sigma}_{ij})$ is an IVIMV between the objects x_i and x_j, and $\overline{\rho}_{ij} = [\rho_{ij}^-, \rho_{ij}^+]$ and $\overline{\sigma}_{ij} = [\sigma_{ij}^-, \sigma_{ij}^+]$ are the interval preferred and non-preferred degrees of the object x_i over x_j, respectively. If the following is true:

$$\begin{cases} \rho_{ij}^- = \sigma_{ji}^-, \rho_{ij}^+ = \sigma_{ji}^+ \\ \sigma_{ij}^- = \rho_{ji}^-, \sigma_{ij}^+ = \rho_{ji}^+ \\ \rho_{ij}^- \leq \rho_{ij}^+, \sigma_{ij}^- \leq \sigma_{ij}^+ \\ 1/9 \leq \rho_{ij}^-, \rho_{ij}^+, \sigma_{ij}^-, \sigma_{ij}^+ \leq 9 \\ \rho_{ij}^+ \sigma_{ij}^+ \leq 1 \\ \rho_{ij}^- = \rho_{ij}^+ = \sigma_{ij}^- = \sigma_{ij}^+ = 1 \end{cases} \quad (5)$$

for all $i, j = 1, 2, \ldots, n$, then $\tilde{R}$ is called an IVIMPR.

According to Definition 7, one can easily find that when $\rho_{ij}^- = \rho_{ij}^+$ and $\sigma_{ij}^- = \sigma_{ij}^+$ for all $i, j = 1, 2, \ldots, n$, the IVIMPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ reduces to an IMPR (*Xia et al., 2013*).

For example, let $X = \{x_1, x_2, x_3\}$. An IVIMPR $\tilde{R}$ on X may be

$$\tilde{R} = \begin{pmatrix} ([1,1],[1,1]) & ([3,4],[\frac{1}{6},\frac{1}{4}]) & ([\frac{1}{2},1],[\frac{1}{3},\frac{1}{2}]) \\ ([\frac{1}{6},\frac{1}{4}],[3,4]) & ([1,1],[1,1]) & ([4,5],[\frac{1}{9},\frac{1}{6}]) \\ ([\frac{1}{3},\frac{1}{2}],[\frac{1}{2},1]) & ([\frac{1}{9},\frac{1}{6}],[4,5]) & ([1,1],[1,1]) \end{pmatrix}.$$

2.2. Two previous consistency concepts

To rank objects from IVIMPRs logically, Zhang and Prdrycz (2019) discussed the consistency of IVIMPRs.

Definition 8 (*Zhang and Prdrycz, 2019*). Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIMPR defined on the object set $X = \{x_1, x_2, \ldots, x_n\}$, where $\tilde{r} = (\overline{\rho}_{ij}, \overline{\sigma}_{ij}) = ([\rho_{ij}^-, \rho_{ij}^+], [\sigma_{ij}^-, \sigma_{ij}^+])$ is an IVIMV for all $i, j = 1, 2, \ldots, n$. Then, $R^- = (r_{ij}^-)_{n \times n}$ and $R^+ = (r_{ij}^+)_{n \times n}$ are separately called the left and right matrices of the IVIMPR $\tilde{R}$, where $r_{ij}^- = \begin{cases} (\rho_{ij}^-, \sigma_{ij}^+) & i < j \\ (0.5, 0.5) & i = j \\ (\rho_{ij}^+, \sigma_{ij}^-) & i > j \end{cases}$ and $r_{ij}^+ = \begin{cases} (\rho_{ij}^+, \sigma_{ij}^-) & i < j \\ (0.5, 0.5) & i = j \\ (\rho_{ij}^-, \sigma_{ij}^+) & i > j \end{cases}$ for all $i, j = 1, 2, \ldots, n$.

According to Definition 8, one can find that the left and right matrices R^- and R^+ are two IMPRs. Based on IMPRs, Zhang and Prdrycz (2019) gave the following consistency concept for IVIMPRs.

Definition 9 (*Zhang and Prdrycz, 2019*). Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIMPR defined on the object set $X = \{x_1, x_2, \ldots, x_n\}$, where $\tilde{r} = (\overline{\rho}_{ij}, \overline{\sigma}_{ij}) = ([\rho_{ij}^-, \rho_{ij}^+], [\sigma_{ij}^-, \sigma_{ij}^+])$ is an IVIMV for all $i, j = 1, 2, \ldots, n$. It is consistent

if its IMPRs $R^- = (r_{ij}^-)_{n \times n}$ and $R^+ = (r_{ij}^+)_{n \times n}$ are both consistent, namely,

$$\rho_{ij}^- \sigma_{ik}^+ \sigma_{kj}^+ = \rho_{ik}^- \rho_{kj}^- \sigma_{ij}^+ \text{ and } \rho_{ij}^+ \sigma_{ik}^- \sigma_{kj}^- = \rho_{ik}^+ \rho_{kj}^+ \sigma_{ij}^- \tag{6}$$

for all $i, k, j = 1, 2, \ldots, n$ with $i < k < j$.

According to Eq. (6), we derive that the IMPRs $R^- = (r_{ij}^-)_{n \times n}$ and $R^+ = (r_{ij}^+)_{n \times n}$ are consistent if and only if $\frac{\rho_{ij}^-}{\sigma_{ij}^+} = \frac{\rho_{ik}^-}{\sigma_{ik}^+} \times \frac{\rho_{kj}^-}{\sigma_{kj}^+}$ and $\frac{\rho_{ij}^+}{\sigma_{ij}^-} = \frac{\rho_{ik}^+}{\sigma_{ik}^-} \times \frac{\rho_{kj}^+}{\sigma_{kj}^-}$ for all $i, k, j = 1, 2, \ldots, n$ with $i < k < j$. It should be noted that this concept is in fact equivalent to Liao and Xu's multiplicative consistency concept for IFPRs (Liao and Xu, 2014). As Meng et al. (2017a) noted, there are two main limitations of Definition 9, namely, it is insufficient to cope with inconsistent and incomplete IVIMPRs which are usually encountered in decision making. Therefore, the application of Definition 9 is very limited. For example, if $\begin{cases} \frac{\rho_{ij}^-}{\sigma_{ij}^+} \neq \frac{\rho_{ik}^-}{\sigma_{ik}^+} \times \frac{\rho_{kj}^-}{\sigma_{kj}^+} \\ \frac{\rho_{ij}^+}{\sigma_{ij}^-} \neq \frac{\rho_{ik}^+}{\sigma_{ik}^-} \times \frac{\rho_{kj}^+}{\sigma_{kj}^-} \end{cases}$ for some triple of (i, k, j), we need to adjust $\frac{\rho_{ij}^-}{\sigma_{ij}^+}$ and $\frac{\rho_{ij}^+}{\sigma_{ij}^-}$ as $\begin{cases} \frac{\rho_{ij}^- + x_{ij}^-}{\sigma_{ij}^+ + y_{ij}^+} = \alpha_{ij} \\ \frac{\rho_{ij}^+ + x_{ij}^+}{\sigma_{ij}^- + y_{ij}^-} = \beta_{ij} \end{cases}$ under the conditions of $\begin{cases} 1/9 \leq \rho_{ij}^- + x_{ij}^- \leq \rho_{ij}^+ + x_{ij}^+ \leq 9 \\ 1/9 \leq \sigma_{ij}^- + y_{ij}^- \leq \sigma_{ij}^+ + y_{ij}^+ \leq 9 \\ (\rho_{ij}^+ + x_{ij}^+) \times (\sigma_{ij}^+ + y_{ij}^+) \leq 1. \end{cases}$ It is difficult to ascertain that the inconsistency is caused by which endpoint of interval preferred and/or non-preferred degrees. For incomplete IVIMPRs, one can check that there may be infinite values for missing judgments satisfying the consistency requirement.

Furthermore, models for calculating the IVIMPWV adopted by Zhang and Prdrycz (2019) cannot ensure the existence of the normalized IVIMPWV even for completely consistent IVIMPRs. For example, let $\tilde{R}$ be an IVIMPR on $X = \{x_1, x_2, x_3\}$, where

$$\tilde{R} = \begin{pmatrix} ([1,1],[1,1]) & ([2,2],[\frac{1}{4},\frac{1}{4}]) & ([1,1],[\frac{1}{8},\frac{1}{8}]) \\ ([\frac{1}{4},\frac{1}{4}],[2,2]) & ([1,1],[1,1]) & ([\frac{1}{4},\frac{1}{4}],[\frac{1}{4},\frac{1}{4}]) \\ ([\frac{1}{8},\frac{1}{8}],[1,1]) & ([\frac{1}{4},\frac{1}{4}],[\frac{1}{4},\frac{1}{4}]) & ([1,1],[1,1]) \end{pmatrix}.$$

One can check that $\tilde{R}$ is completely consistent according to Definition 9. According to Eqs. (30) and (31) in the literature (Zhang and Prdrycz, 2019), we obtain

$$\bar{W} = \left(\left(\left[\frac{1}{w_1^{\sigma-}}, \frac{1}{w_1^{\sigma-}} \right], [w_1^{\sigma-}, w_1^{\sigma-}] \right), \right.$$
$$\left(\left[\frac{1}{4w_1^{\sigma-}}, \frac{1}{4w_1^{\sigma-}} \right], [2w_1^{\sigma-}, 2w_1^{\sigma-}] \right),$$
$$\left. \left(\left[\frac{1}{8w_1^{\sigma-}}, \frac{1}{8w_1^{\sigma-}} \right], [w_1^{\sigma-}, w_1^{\sigma-}] \right) \right),$$

where $w_1^{\sigma-}$ is the lower bound of the interval non-preferred degree of the object x_1.

According to the definition of the normalized IVIMPWV, we should have

$$\begin{cases} w_1^{\rho-} w_2^{\rho-} \leq w_3^{\sigma-} \\ w_3^{\rho-} \geq w_1^{\sigma-} w_2^{\sigma-} \end{cases} \Rightarrow \begin{cases} \frac{1}{w_1^{\sigma-}} \times \frac{1}{4w_1^{\sigma-}} \leq w_1^{\sigma-} \\ \frac{1}{8w_1^{\sigma-}} \geq w_1^{\sigma-} \times 2w_1^{\sigma-} \end{cases} \Rightarrow \begin{cases} \sqrt[3]{\frac{1}{4}} \leq w_1^{\sigma-} \\ \sqrt[3]{\frac{1}{16}} \geq w_1^{\sigma-}. \end{cases}$$

Thus, Definitions 4 and 6 given in the literature (Zhang and Prdrycz, 2019) are not equivalent. The rationality of models (M-9) to (M-11) in the literature (Zhang and Prdrycz, 2019) for calculating the normalized IVIMPWV is questionable.

On the other hand, Sahu et al. (2018) extended Liu's consistency concept for IVMPRs (Liu, 2009) to offer another consistency concept for IVIMPRs.

Definition 10 (*Sahu et al., 2018*). Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIMPR defined on the object set $X = \{x_1, x_2, \ldots, x_n\}$, where $\tilde{r} = (\bar{\rho}_{ij}, \bar{\sigma}_{ij}) = \left([\rho_{ij}^-, \rho_{ij}^+], [\sigma_{ij}^-, \sigma_{ij}^+] \right)$ is an IVIMV for all $i, j = 1, 2, \ldots, n$. It is consistent if its associated four multiplicative preference relations (MPRs) $B^{(q)} = \left(b_{ij}^{(q)} \right)_{n \times n}$, $q = 1, 2, 3, 4$, are all consistent, namely, $b_{ij}^{(q)} = b_{ik}^{(q)} b_{kj}^{(q)}$ for all $i, j = 1, 2, \ldots, n$, where

$$b_{ij}^{(1)} = \begin{cases} \rho_{ij}^- & i < j \\ 1 & i = j \\ 1/\rho_{ij}^- & i > j \end{cases}, \ b_{ij}^{(2)} = \begin{cases} \rho_{ij}^+ & i < j \\ 1 & i = j \\ 1/\rho_{ij}^+ & i > j \end{cases}, \ b_{ij}^{(3)} = \begin{cases} \sigma_{ij}^- & i < j \\ 1 & i = j \\ 1/\sigma_{ij}^- & i > j \end{cases}, \text{ and }$$

$$b_{ij}^{(4)} = \begin{cases} \sigma_{ij}^+ & i < j \\ 1 & i = j \\ 1/\sigma_{ij}^+ & i > j \end{cases} \tag{7}$$

Considering the fact that the principle of Definition 10 is similar to that offered by Liu (2009) and Jiang et al. (2015), one can easily check that Definition 10 has the limitations listed in (2) in Introduction. Taking the first issue for example, let $\tilde{R}$ be an IVIMPR on $X = \{x_1, x_2, x_3\}$, where

$$\tilde{R} = \begin{pmatrix} ([1,1],[1,1]) & ([\frac{1}{2},\frac{1}{4}],[2,3]) & ([\frac{1}{2},\frac{1}{2}],[\frac{2}{3},\frac{3}{4}]) \\ ([2,3],[\frac{1}{2},\frac{1}{4}]) & ([1,1],[1,1]) & ([1,2],[\frac{1}{3},\frac{1}{4}]) \\ ([\frac{2}{3},\frac{3}{4}],[\frac{1}{2},\frac{1}{2}]) & ([\frac{1}{3},\frac{1}{4}],[1,2]) & ([1,1],[1,1]) \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3. \end{matrix}$$

According to Eq. (7), we have $B^{(1)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{2} \\ 4 & 1 & 2 \\ 2 & \frac{1}{2} & 1 \end{pmatrix}$, $B^{(3)} = \begin{pmatrix} 1 & 2 & \frac{2}{3} \\ \frac{1}{2} & 1 & \frac{1}{3} \\ \frac{3}{2} & 3 & 1 \end{pmatrix}$, and $B^{(4)} = \begin{pmatrix} 1 & 3 & \frac{3}{4} \\ \frac{1}{3} & 1 & \frac{1}{4} \\ \frac{4}{3} & 4 & 1 \end{pmatrix}$. Because all of these MPRs are consistent, $\tilde{R}$ is consistent. In addition, let σ be a permutation on X, where $x_{\sigma(1)} = x_3$, $x_{\sigma(2)} = x_1$, and $x_{\sigma(3)} = x_2$. Then,

$$\tilde{R}^\sigma = \begin{pmatrix} ([1,1],[1,1]) & ([1,2],[\frac{1}{3},\frac{1}{4}]) & ([2,3],[\frac{1}{2},\frac{1}{4}]) \\ ([\frac{1}{3},\frac{1}{4}],[1,2]) & ([1,1],[1,1]) & ([\frac{2}{3},\frac{3}{4}],[\frac{1}{2},\frac{1}{2}]) \\ ([\frac{1}{2},\frac{1}{4}],[2,3]) & ([\frac{1}{2},\frac{1}{2}],[\frac{2}{3},\frac{3}{4}]) & ([1,1],[1,1]) \end{pmatrix} \begin{matrix} x_2 \\ x_3 \\ x_1. \end{matrix}$$

With respect to $\tilde{R}^\sigma$, we obtain $B^{\sigma(1)} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} & 1 \end{pmatrix}$, $B^{\sigma(2)} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1 & \frac{3}{4} \\ \frac{1}{3} & \frac{4}{3} & 1 \end{pmatrix}$, $B^{\sigma(3)} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & \frac{1}{2} \\ 2 & 2 & 1 \end{pmatrix}$, and $B^{\sigma(4)} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ 4 & 1 & \frac{1}{2} \\ 4 & 2 & 1 \end{pmatrix}$. One can check that none of these MPRs $B^{\sigma(q)}$, $q = 1, 2, 3, 4$, is consistent. Hence, $\tilde{R}^\sigma$ is inconsistent.

3. A new consistency concept

To obtain the ranking rationally, consistency analysis should be conducted. Considering the limitations of previous consistency concepts for IVIMPRs, we next study the consistency of IVIMPRs. As analysis in Ref. Meng et al. (2017b), it is unsuitable to directly apply the consistency concept for crisp preference relations to define the consistency of IVIM-PRs. Considering this, we introduce the following concept of preferred 2-dimensional interval fuzzy multiplicative variables (PTDIFMVs):

Definition 11. Let $\tilde{\alpha} = ([\rho^-, \rho^+], [\sigma^-, \sigma^+])$ be an IVIMV. Then, $\tilde{\alpha} = ([\rho^-, \rho^+], [1/\sigma^+, 1/\sigma^-])$ is called a PTDIFMV.

Let $\tilde{\alpha} = ([\rho^-, \rho^+], [\sigma^-, \sigma^+])$. Because $[\sigma^-, \sigma^+]$ denotes the non-preferred interval multiplicative membership degree, $[1/\sigma^+, 1/\sigma^-]$ is the preferred interval multiplicative membership degree for $[\sigma^-, \sigma^+]$.

Definition 12. Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIMPR, where $\tilde{r}_{ij} = (\bar{\rho}_{ij}, \bar{\sigma}_{ij})$ is an IVIMV. Then, $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is called a preferred 2-dimensional interval fuzzy multiplicative preference relation (PTDIFMPR), where $\tilde{p}_{ij} = \left(\bar{\rho}_{ij}, \bar{\sigma}_{ij}^{-1}\right) = ([\rho_{ij}^-, \rho_{ij}^+], [1/\sigma_{ij}^+, 1/\sigma_{ij}^-])$ is a PTDIFMV for the IVIMV $\tilde{r}_{ij}$, $i, j = 1, 2, \ldots, n$.

Definition 12 shows that a PTDIFMPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ corresponds to two IVMPRs $\bar{B}_1 = (\bar{b}_{1,ij})_{n\times n}$ and $\bar{B}_2 = (\bar{b}_{2,ij})_{n\times n}$, where

$$\bar{b}_{1,ij} = \begin{cases} [\rho_{ij}^-, \rho_{ij}^+] & i < j \\ [1,1] & i = j \\ [1/\rho_{ij}^+, 1/\rho_{ij}^-] & i > j \end{cases} \text{ and } \bar{b}_{2,ij} = \begin{cases} [1/\sigma_{ij}^+, 1/\sigma_{ij}^-] & i < j \\ [1,1] & i = j \\ [\sigma_{ij}^-, \sigma_{ij}^+] & i > j \end{cases} \quad (8)$$

Definition 13. Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIMPR, and let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be its associated PTDIFMPR. $\tilde{P}$ is consistent if and only if the corresponding two IVMPRs $\bar{B}_1 = (\bar{b}_{1,ij})_{n\times n}$ and $\bar{B}_2 = (\bar{b}_{2,ij})_{n\times n}$ defined in Eq. (8) are both consistent based on Definition 4.

Definition 12 shows that elements in PTDIFMPRs are derived from associated IVIMPRs. Thus, we can apply consistent PTDIFMPRs to define the consistency of IVIMPRs.

Definition 14. Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIMPR. It is consistent if one of its associated PTDIFMPRs shown in Definition 12 is consistent based on Definition 13.

Following discussion about the properties of Definition 4 (Meng and Tan, 2017), one can check that Definition 14 satisfies two important properties for consistency concepts: robustness and upper triangular property.

As Meng and Tan (2017) noted, it is infeasible to directly apply Definition 14 to judge the consistency of IVIMPRs because there are too many QIVMPRs. Considering this situation, we build the following model to judge the consistency of IVIMPRs.

Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIMPR. If it is consistent, then the IVMPRs $\bar{B}_1 = (\bar{b}_{1,ij})_{n\times n}$ and $\bar{B}_2 = (\bar{b}_{2,ij})_{n\times n}$ obtained from the PTDIFMPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ are consistent simultaneously. Following Definition 4, consistent QIVMPRs exist for the IVMPRs $\bar{B}_1 = (\bar{b}_{1,ij})_{n\times n}$ and $\bar{B}_2 = (\bar{b}_{2,ij})_{n\times n}$. Without loss of generality, suppose that $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ and $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ are their corresponding consistent QIVMPRs, where

$$\begin{cases} \bar{q}_{1,ij} = (\bar{b}_{1,ij})^{\theta_{ij}} \otimes (\bar{b}_{1,ij}^\circ)^{1-\theta_{ij}} \\ \bar{q}_{1,ji} = (\bar{b}_{1,ji}^\circ)^{\theta_{ji}} \otimes (\bar{b}_{1,ji})^{1-\theta_{ji}} \end{cases} \text{ and } \begin{cases} \bar{q}_{2,ij} = (\bar{b}_{2,ij})^{\vartheta_{ij}} \otimes (\bar{b}_{2,ij}^\circ)^{1-\vartheta_{ij}} \\ \bar{q}_{2,ji} = (\bar{b}_{2,ji}^\circ)^{\vartheta_{ji}} \otimes (\bar{b}_{2,ji})^{1-\vartheta_{ji}}, \end{cases} \quad (9)$$

θ_{ij} and ϑ_{ij} are the 0-1-IVs such that $\begin{cases} \theta_{ij} + \theta_{ji} = 1 \\ \vartheta_{ij} + \vartheta_{ji} = 1 \end{cases}$ defined as: $\theta_{ij} = \begin{cases} 1 & \text{if } \bar{q}_{1,ij} = \bar{b}_{1,ij} \text{ and } \bar{q}_{1,ji} = \bar{b}_{1,ji}^\circ \\ 0 & \text{if } \bar{q}_{1,ij} = \bar{b}_{1,ij}^\circ \text{ and } \bar{q}_{1,ji} = \bar{b}_{1,ji} \end{cases}$ and $\vartheta_{ij} = \begin{cases} 1 & \text{if } \bar{q}_{2,ij} = \bar{b}_{2,ij} \text{ and } \bar{q}_{2,ji} = \bar{b}_{2,ji}^\circ \\ 0 & \text{if } \bar{q}_{2,ij} = \bar{b}_{2,ij}^\circ \text{ and } \bar{q}_{2,ji} = \bar{b}_{2,ji} \end{cases}$ for all $i, j = 1, 2, \ldots, n$.

Then, Definition 4 shows that the following is true:

$$\begin{cases} \bar{q}_{1,ij} = \bar{q}_{1,ik} \otimes \bar{q}_{1,kj} \\ \bar{q}_{2,ij} = \bar{q}_{2,ik} \otimes \bar{q}_{2,kj} \end{cases} \quad (10)$$

where $i, k, j = 1, 2, \ldots, n$.

From Eqs. (9) and (10), we obtain

$$\begin{cases} \left((\bar{b}_{1,ij})^{\theta_{ij}} \otimes (\bar{b}_{1,ij}^\circ)^{1-\theta_{ij}}\right) = \left((\bar{b}_{1,ik})^{\theta_{ik}} \otimes (\bar{b}_{1,ik}^\circ)^{1-\theta_{ik}}\right) \otimes \left((\bar{b}_{1,kj})^{\theta_{kj}} \otimes (\bar{b}_{1,kj}^\circ)^{1-\theta_{kj}}\right) \\ \left((\bar{b}_{2,ij})^{\vartheta_{ij}} \otimes (\bar{b}_{2,ij}^\circ)^{1-\vartheta_{ij}}\right) = \left((\bar{b}_{2,ik})^{\vartheta_{ik}} \otimes (\bar{b}_{2,ik}^\circ)^{1-\vartheta_{ik}}\right) \otimes \left((\bar{b}_{2,kj})^{\vartheta_{kj}} \otimes (\bar{b}_{2,kj}^\circ)^{1-\vartheta_{kj}}\right) \end{cases} \quad (11)$$

where $i, k, j = 1, 2, \ldots, n$.

From the upper triangular property and Eq. (8), we get

$$\begin{cases} \left([\rho_{ij}^-, \rho_{ij}^+]^{\theta_{ij}} \otimes [\rho_{ij}^+, \rho_{ij}^-]^{1-\theta_{ij}}\right) = \left([\rho_{ik}^-, \rho_{ik}^+]^{\theta_{ik}} \otimes [\rho_{ik}^+, \rho_{ik}^-]^{1-\theta_{ik}}\right) \\ \otimes \left([\rho_{kj}^-, \rho_{kj}^+]^{\theta_{kj}} \otimes [\rho_{kj}^+, \rho_{kj}^-]^{1-\theta_{kj}}\right), i, k, j = 1, 2, \ldots, n; i < k < j \\ \left([\sigma_{ij}^-, \sigma_{ij}^+]^{\vartheta_{ij}} \otimes [\sigma_{ij}^+, \sigma_{ij}^-]^{1-\vartheta_{ij}}\right) = \left([\sigma_{ik}^-, \sigma_{ik}^+]^{\vartheta_{ik}} \otimes [\sigma_{ik}^+, \sigma_{ik}^-]^{1-\vartheta_{ik}}\right) \\ \otimes \left([\sigma_{kj}^-, \sigma_{kj}^+]^{\vartheta_{kj}} \otimes [\sigma_{kj}^+, \sigma_{kj}^-]^{1-\vartheta_{kj}}\right), i, k, j = 1, 2, \ldots, n; i > k > j \end{cases} \quad (12)$$

Remark 1. Note that we apply the upper triangular part of the QIVMPR $\bar{Q}_1 = (\bar{q}_{1,ij})_{n\times n}$ to define its consistency, while the lower triangular part of the QIVMPR $\bar{Q}_2 = (\bar{q}_{2,ij})_{n\times n}$ is adopted to define the consistency of $\bar{Q}_2$, which are composed by the preferred and non-preferred multiplicative interval membership degrees of IVIMVs in $\tilde{R}$, respectively.

We take the logarithm on Eq. (12) and derive

$$\begin{cases} \theta_{ij}[\log(\rho_{ij}^-), \log(\rho_{ij}^+)] + (1-\theta_{ij})[\log(\rho_{ij}^+), \log(\rho_{ij}^-)] \\ = \left(\theta_{ik}[\log(\rho_{ik}^-), \log(\rho_{ik}^+)] + (1-\theta_{ik})[\log(\rho_{ik}^+), \log(\rho_{ik}^-)]\right) \\ \oplus \left(\theta_{kj}[\log(\rho_{kj}^-), \log(\rho_{kj}^+)] + (1-\theta_{kj})[\log(\rho_{kj}^+), \log(\rho_{kj}^-)]\right), \\ i, k, j = 1, 2, \ldots, n; i < k < j \\ \vartheta_{ij}[\log(\sigma_{ij}^-), \log(\sigma_{ij}^+)] + (1-\vartheta_{ij})[\log(\sigma_{ij}^+), \log(\sigma_{ij}^-)] \\ = \left(\vartheta_{ik}[\log(\sigma_{ik}^-), \log(\sigma_{ik}^+)] + (1-\vartheta_{ik})[\log(\sigma_{ik}^+), \log(\sigma_{ik}^-)]\right) \\ \oplus \left(\vartheta_{kj}[\log(\sigma_{kj}^-), \log(\sigma_{kj}^+)] + (1-\vartheta_{kj})[\log(\sigma_{kj}^+), \log(\sigma_{kj}^-)]\right), \\ i, k, j = 1, 2, \ldots, n; i > k > j \end{cases} \quad (13)$$

To judge whether Eq. (13) is true, we build the following two models:

$$\varphi_1^* = \min \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \left(\chi_{k,ij}^+ + \chi_{k,ij}^- + \gamma_{k,ij}^+ + \gamma_{k,ij}^-\right)$$

$$s.t. \begin{cases} \theta_{ik}\log(\rho_{ik}^-) + (1-\theta_{ik})\log(\rho_{ik}^+) + \theta_{kj}\log(\rho_{kj}^-) + (1-\theta_{kj})\log(\rho_{kj}^+) \\ = \theta_{ij}\log(\rho_{ij}^-) + (1-\theta_{ij})\log(\rho_{ij}^+) - \chi_{k,ij}^+ + \chi_{k,ij}^-, i, k, j = 1, 2, \ldots, n; i < k < j \\ \theta_{ik}\log(\rho_{ik}^+) + (1-\theta_{ik})\log(\rho_{ik}^-) + \theta_{kj}\log(\rho_{kj}^+) + (1-\theta_{kj})\log(\rho_{kj}^-) \\ = \theta_{ij}\log(\rho_{ij}^+) + (1-\theta_{ij})\log(\rho_{ij}^-) - \gamma_{k,ij}^+ + \gamma_{k,ij}^-, i, k, j = 1, 2, \ldots, n; i < k < j \\ \chi_{k,ij}^+, \chi_{k,ij}^-, \gamma_{k,ij}^+, \gamma_{k,ij}^- \geq 0, i, k, j = 1, 2, \ldots, n; i < k < j \\ \theta_{ij} = 0 \vee 1, i, j = 1, 2, \ldots, n; i < j \end{cases}$$

(M-1)

and

$$\varphi_2^* = \min \sum_{i=k+1}^{n} \sum_{k=j+1}^{n-1} \sum_{j=1}^{n-2} \left(\eta_{k,ij}^+ + \eta_{k,ij}^- + \mu_{k,ij}^+ + \mu_{k,ij}^-\right)$$

$$s.t. \begin{cases} \vartheta_{ik}\log(\sigma_{ik}^-) + (1-\vartheta_{ik})\log(\sigma_{ik}^+) + \vartheta_{kj}\log(\sigma_{kj}^-) + (1-\vartheta_{kj})\log(\sigma_{kj}^+) \\ = \vartheta_{ij}\log(\sigma_{ij}^-) + (1-\vartheta_{ij})\log(\sigma_{ij}^+) - \eta_{k,ij}^+ + \eta_{k,ij}^-, i, k, j = 1, 2, \ldots, n; i > k > j \\ \vartheta_{ik}\log(\sigma_{ik}^+) + (1-\vartheta_{ik})\log(\sigma_{ik}^-) + \vartheta_{kj}\log(\sigma_{kj}^+) + (1-\vartheta_{kj})\log(\sigma_{kj}^-) \\ = \vartheta_{ij}\log(\sigma_{ij}^+) + (1-\vartheta_{ij})\log(\sigma_{ij}^-) - \mu_{k,ij}^+ + \mu_{k,ij}^-, i, k, j = 1, 2, \ldots, n; i > k > j \\ \eta_{k,ij}^+, \eta_{k,ij}^-, \mu_{k,ij}^+, \mu_{k,ij}^- \geq 0, i, k, j = 1, 2, \ldots, n; i > k > j \\ \vartheta_{ij} = 0 \vee 1, i, j = 1, 2, \ldots, n; i > j \end{cases}$$

(M-2)

where $i, k, j = 1, 2, \ldots, n$.

where the constraints in models (M-1) and (M-2) are separately derived from the first and second equations in (13) by adding non-negative flexible variables $\chi_{k,ij}^+$, $\chi_{k,ij}^-$, $\gamma_{k,ij}^+$, $\gamma_{k,ij}^-$, $\eta_{k,ij}^+$, $\eta_{k,ij}^-$, $\mu_{k,ij}^+$, and $\mu_{k,ij}^-$ for each triple of (i, k, j) such that $i < k < j$, θ_{ij} and ϑ_{ij} are 0–1 indicator variables as shown in Eq. (9).

Models (M-1) and (M-2) can be equivalently combined into the following model:

$$\varphi^* = \min \left(\sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \left(\chi_{k,ij}^+ + \chi_{k,ij}^- + \gamma_{k,ij}^+ + \gamma_{k,ij}^- \right) \right.$$
$$\left. + \sum_{i=k+1}^{n} \sum_{k=j+1}^{n-1} \sum_{j=1}^{n-2} \left(\eta_{k,ij}^+ + \eta_{k,ij}^- + \mu_{k,ij}^+ + \mu_{k,ij}^- \right) \right)$$

$$s.t. \begin{cases} \theta_{ik} \log(\rho_{ik}^-) + (1 - \theta_{ik}) \log(\rho_{ik}^+) + \theta_{kj} \log(\rho_{kj}^-) + (1 - \theta_{kj}) \log(\rho_{kj}^+) \\ = \theta_{ij} \log(\rho_{ij}^-) + (1 - \theta_{ij}) \log(\rho_{ij}^+) - \chi_{k,ij}^+ + \chi_{k,ij}^-, i, k, j = 1, 2, \dots, n; i < k < j \\ \theta_{ik} \log(\rho_{ik}^+) + (1 - \theta_{ik}) \log(\rho_{ik}^-) + \theta_{kj} \log(\rho_{kj}^+) + (1 - \theta_{kj}) \log(\rho_{kj}^-) \\ = \theta_{ij} \log(\rho_{ij}^+) + (1 - \theta_{ij}) \log(\rho_{ij}^-) - \gamma_{k,ij}^+ + \gamma_{k,ij}^-, i, k, j = 1, 2, \dots, n; i < k < j \\ \chi_{k,ij}^+, \chi_{k,ij}^-, \gamma_{k,ij}^+, \gamma_{k,ij}^- \geq 0, i, k, j = 1, 2, \dots, n; i < k < j \\ \theta_{ij} = 0 \vee 1, i, j = 1, 2, \dots, n; i < j \\ \vartheta_{ik} \log(\sigma_{ik}^-) + (1 - \vartheta_{ik}) \log(\sigma_{ik}^+) + \vartheta_{kj} \log(\sigma_{kj}^-) + (1 - \vartheta_{kj}) \log(\sigma_{kj}^+) \\ = \vartheta_{ij} \log(\sigma_{ij}^-) + (1 - \vartheta_{ij}) \log(\sigma_{ij}^+) - \eta_{k,ij}^+ + \eta_{k,ij}^-, i, k, j = 1, 2, \dots, n; i > k > j \\ \vartheta_{ik} \log(\sigma_{ik}^+) + (1 - \vartheta_{ik}) \log(\sigma_{ik}^-) + \vartheta_{kj} \log(\sigma_{kj}^+) + (1 - \vartheta_{kj}) \log(\sigma_{kj}^-) \\ = \vartheta_{ij} \log(\sigma_{ij}^+) + (1 - \vartheta_{ij}) \log(\sigma_{ij}^-) - \tau_{k,ij}^+ + \tau_{k,ij}^-, i, k, j = 1, 2, \dots, n; i > k > j \\ \eta_{k,ij}^+, \eta_{k,ij}^-, \tau_{k,ij}^+, \tau_{k,ij}^- \geq 0, i, k, j = 1, 2, \dots, n; i > k > j \\ \vartheta_{ij} = 0 \vee 1, i, j = 1, 2, \dots, n; i > j \end{cases}$$

(M-3)

where the objective function and the constraints in model (M-3) are derived from those in models (M-1) and (M-2).

Addressing model (M-3), if $\varphi^* = 0$, then $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ is consistent. Based on the got optimal 0-1-IVs, we can obtain the associated consistent QIVMPRs.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$ be the object set. The IVIMPR $\tilde{R}$ on X is defined as:

$$\tilde{R} = \begin{pmatrix} ([1,1],[1,1]) & ([2,3],[\frac{1}{4},\frac{1}{3}]) & ([1,1],[\frac{1}{3},\frac{1}{2}]) & ([\frac{1}{2},1],[\frac{1}{8},\frac{1}{4}]) \\ ([\frac{1}{4},\frac{1}{3}],[2,3]) & ([1,1],[1,1]) & ([\frac{1}{3},\frac{1}{2}],[1,2]) & ([\frac{1}{4},\frac{1}{3}],[\frac{3}{8},1]) \\ ([\frac{1}{3},\frac{1}{2}],[1,1]) & ([1,2],[\frac{1}{3},\frac{1}{2}]) & ([1,1],[1,1]) & ([\frac{1}{2},1],[\frac{3}{8},\frac{1}{2}]) \\ ([\frac{1}{8},\frac{1}{4}],[\frac{1}{2},1]) & ([\frac{3}{8},1],[\frac{1}{4},\frac{1}{3}]) & ([\frac{3}{8},\frac{1}{2}],[\frac{1}{2},1]) & ([1,1],[1,1]) \end{pmatrix}.$$

Following $\tilde{R}$, the PTDIFMPR $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ is

$$\tilde{P} = \begin{pmatrix} ([1,1],[1,1]) & ([2,3],[3,4]) & ([1,1],[2,3]) & ([\frac{1}{2},1],[4,8]) \\ ([\frac{1}{4},\frac{1}{3}],[\frac{1}{3},\frac{1}{2}]) & ([1,1],[1,1]) & ([\frac{1}{3},\frac{1}{2}],[\frac{1}{2},1]) & ([\frac{1}{4},\frac{1}{3}],[1,\frac{8}{3}]) \\ ([\frac{1}{3},\frac{1}{2}],[1,1]) & ([\frac{1}{2},1],[2,3]) & ([1,1],[1,1]) & ([\frac{1}{2},1],[2,\frac{8}{3}]) \\ ([\frac{1}{8},\frac{1}{4}],[1,2]) & ([\frac{3}{8},1],[3,4]) & ([\frac{3}{8},\frac{1}{2}],[1,2]) & ([1,1],[1,1]) \end{pmatrix}.$$

Furthermore, two associated IVMPRs are

$$\overline{B}_1 = \begin{pmatrix} [1,1] & [2,3] & [1,1] & [\frac{1}{2},1] \\ [\frac{1}{3},\frac{1}{2}] & [1,1] & [\frac{1}{3},\frac{1}{2}] & [\frac{1}{4},\frac{1}{3}] \\ [1,1] & [2,3] & [1,1] & [\frac{1}{2},1] \\ [1,2] & [3,4] & [1,2] & [1,1] \end{pmatrix} \text{ and }$$

$$\overline{B}_2 = \begin{pmatrix} [1,1] & [3,4] & [2,3] & [4,8] \\ [\frac{1}{4},\frac{1}{3}] & [1,1] & [\frac{1}{2},1] & [1,\frac{8}{3}] \\ [\frac{1}{3},\frac{1}{2}] & [1,2] & [1,1] & [2,\frac{8}{3}] \\ [\frac{1}{8},\frac{1}{4}] & [\frac{3}{8},1] & [\frac{3}{8},\frac{1}{2}] & [1,1] \end{pmatrix}.$$

Using model (M-3), we have $\varphi^* = 0$. Thus, this IVIMPR $\tilde{R}$ is consistent following Definition 14. Based on

$$\begin{cases} \theta_{23}^* = 1, \theta_{12}^* = \theta_{13}^* = \theta_{14}^* = \theta_{24}^* = \theta_{34}^* = 0 \\ \vartheta_{21}^* = 0, \vartheta_{31}^* = \vartheta_{32}^* = \vartheta_{41}^* = \vartheta_{42}^* = \vartheta_{43}^* = 1 \end{cases} \text{ derived from model (M-3),}$$

the consistent QIVMPRs are

$$\overline{Q}_1 = \begin{pmatrix} [1,1] & [3,2] & [1,1] & [1,\frac{1}{2}] \\ [\frac{1}{3},\frac{1}{2}] & [1,1] & [\frac{1}{3},\frac{1}{2}] & [\frac{1}{4},\frac{1}{3}] \\ [1,1] & [3,2] & [1,1] & [1,\frac{1}{2}] \\ [1,2] & [3,4] & [1,2] & [1,1] \end{pmatrix} \text{ and }$$

$$\overline{Q}_2 = \begin{pmatrix} [1,1] & [3,4] & [3,2] & [8,4] \\ [\frac{1}{3},\frac{1}{4}] & [1,1] & [1,\frac{1}{2}] & [\frac{8}{3},1] \\ [\frac{1}{3},\frac{1}{2}] & [1,2] & [1,1] & [\frac{8}{3},2] \\ [\frac{1}{8},\frac{1}{4}] & [\frac{3}{8},1] & [\frac{3}{8},\frac{1}{2}] & [1,1] \end{pmatrix}.$$

One can check that this IVIMPR $\tilde{R}$ is inconsistent following Definitions 9 and 10.

4. Incomplete and inconsistent IVIMPRs

Incomplete and inconsistent preference relations are usually obtained in decision making (Capuano et al., 2018). This section studies incomplete and inconsistent IVIMPRs. The first part studies incomplete IVIMPRs, and the second part discusses inconsistent IVIMPRs.

4.1. A model to get missing values

In many situations, because of the limited expertise and the complexity of decision-making problems, some judgements may be missing. Therefore, we can only derive incomplete preference relations. This subsection focuses on incomplete IVIMPRs, namely, there are unknown values in IVIMPRs.

For any incomplete IVIMPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, let

$$\begin{cases} U_1 = \{\rho_{ij}^- \text{ is missing, where } i, j = 1, 2, \dots, n, i < j\} \\ U_2 = \{\rho_{ij}^+ \text{ is missing, where } i, j = 1, 2, \dots, n, i < j\} \end{cases}$$

and

$$\begin{cases} U_3 = \{\sigma_{ij}^- \text{ is missing, where } i, j = 1, 2, \dots, n, i < j\} \\ U_4 = \{\sigma_{ij}^+ \text{ is missing, where } i, j = 1, 2, \dots, n, i < j\}. \end{cases}$$

When there are values in [1/9, 9] for all unknown values in $\tilde{R}$ that make it consistent, Eq. (10) holds. Next, we consider another equivalent

consistency condition to address the case where ignored objects exist, namely, all information for some objects is unknown.

Property 1. *Let $\bar{R} = (\tilde{r}_{ij})_{n \times n}$ be an (incomplete) IVIMPR. Then, it is consistent if and only if the following is true:*

$$
\begin{cases}
\left(\overline{q}_{1,ij}\right)^{n-2} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right) \\
\left(\overline{q}_{2,ij}\right)^{n-2} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right),
\end{cases}
\tag{14}
$$

where $\overline{Q}_1 = (\overline{q}_{1,ij})_{n \times n}$ and $\overline{Q}_2 = (\overline{q}_{2,ij})_{n \times n}$ are the QIVMPRs for the IVMPRs $\overline{B}_1 = (\overline{b}_{1,ij})_{n \times n}$ and $\overline{B}_2 = (\overline{b}_{2,ij})_{n \times n}$ obtained from the PTDIFMPR $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$, respectively.

Proof. When the (incomplete) IVIMPR $\bar{R} = (\tilde{r}_{ij})_{\underline{n \times n}}$ is consistent, the QIVMPRs $\overline{Q}_1 = (\overline{q}_{1,ij})_{n \times n} = ([q_{1,ij}^-, q_{1,ij}^+])_{n \times n}$ and $\overline{Q}_2 = (\overline{q}_{2,ij})_{n \times n} = ([q_{2,ij}^-, q_{2,ij}^+])_{n \times n}$ for the IVMPRs $\overline{B}_1 = (\overline{b}_{1,ij})_{n \times n}$ and $\overline{B}_2 = (\overline{b}_{2,ij})_{n \times n}$ are consistent, namely, Eq. (10) is true for $\overline{Q}_1$ and $\overline{Q}_2$. From Eq. (10), we get

$$
\begin{cases}
\left(\overline{q}_{1,ij}\right)^{n} = \otimes_{k=1}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right) \\
\left(\overline{q}_{2,ij}\right)^{n} = \otimes_{k=1}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right)
\end{cases}
\tag{15}
$$

where $i, j = 1, 2, \ldots, n$.

For per Eq. (15), we obtain

$$
\begin{cases}
\left(\overline{q}_{1,ij}\right)^{n} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right) \otimes \left(\overline{q}_{1,ii} \otimes \overline{q}_{1,ij}\right) \otimes \left(\overline{q}_{1,ij} \otimes \overline{q}_{1,jj}\right) \\
\left(\overline{q}_{2,ij}\right)^{n} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right) \otimes \left(\overline{q}_{2,ii} \otimes \overline{q}_{2,ij}\right) \otimes \left(\overline{q}_{2,ij} \otimes \overline{q}_{2,jj}\right)
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
\left(\overline{q}_{1,ij}\right)^{n} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right) \otimes \left(\overline{q}_{1,ij}\right)^{2} \\
\left(\overline{q}_{2,ij}\right)^{n} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right) \otimes \left(\overline{q}_{2,ij}\right)^{2}
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
\left(\overline{q}_{1,ij}\right)^{n} \otimes [1/q_{1,ij}^-, 1/q_{1,ij}^+]^2 \\
\quad = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right) \otimes \left(\overline{q}_{1,ij}\right)^2 \otimes [1/q_{1,ij}^-, 1/q_{1,ij}^+]^2 \\
\left(\overline{q}_{2,ij}\right)^{n} \otimes [1/q_{2,ij}^-, 1/q_{2,ij}^+]^2 \\
\quad = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right) \otimes \left(\overline{q}_{2,ij}\right)^2 \otimes [1/q_{2,ij}^-, 1/q_{2,ij}^+]^2
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
\left(\overline{q}_{1,ij}\right)^{n-2} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right) \\
\left(\overline{q}_{2,ij}\right)^{n-2} = \otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right).
\end{cases}
$$

When Eq. (15) holds, we have

$$
\begin{cases}
\overline{q}_{1,ij} = \sqrt[n-2]{\otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kj}\right)} \\
\overline{q}_{2,ij} = \sqrt[n-2]{\otimes_{k=1, k \neq i, j}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kj}\right)}
\end{cases}
\tag{16}
$$

Eq. (16) shows that

$$
\begin{cases}
\overline{q}_{1,il} \otimes \overline{q}_{1,lj} = \sqrt[n-2]{\otimes_{k=1, k \neq i, l}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kl}\right)} \\
\qquad \otimes \sqrt[n-2]{\otimes_{k=1, k \neq l, j}^{n} \left(\overline{q}_{1,lk} \otimes \overline{q}_{1,kj}\right)} \\
\overline{q}_{2,il} \otimes \overline{q}_{2,lj} = \sqrt[n-2]{\otimes_{k=1, k \neq i, l}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kl}\right)} \\
\qquad \otimes \sqrt[n-2]{\otimes_{k=1, k \neq l, j}^{n} \left(\overline{q}_{2,lk} \otimes \overline{q}_{2,kj}\right)}
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
\overline{q}_{1,il} \otimes \overline{q}_{1,lj} = \sqrt[n-2]{\otimes_{k=1, k \neq i, l}^{n} \left(\overline{q}_{1,ik} \otimes \overline{q}_{1,kl} \otimes \overline{q}_{1,lk} \otimes \overline{q}_{1,kj}\right)} \\
\overline{q}_{2,il} \otimes \overline{q}_{2,lj} = \sqrt[n-2]{\otimes_{k=1, k \neq i, l}^{n} \left(\overline{q}_{2,ik} \otimes \overline{q}_{2,kl} \otimes \overline{q}_{2,lk} \otimes \overline{q}_{2,kj}\right)}
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
\overline{b}_{1,il} \otimes \overline{b}_{1,lj} = \sqrt[n-2]{\otimes_{k=1, k \neq i, l}^{n} \left(\overline{b}_{1,ik} \otimes \overline{b}_{1,kj}\right)} = \overline{b}_{1,ij} \\
\overline{b}_{2,il} \otimes \overline{b}_{2,lj} = \sqrt[n-2]{\otimes_{k=1, k \neq i, l}^{n} \left(\overline{b}_{2,ik} \otimes \overline{b}_{2,kj}\right)} = \overline{b}_{2,ij}
\end{cases}
$$

When an incomplete IVIMPR $\bar{R} = (\tilde{r}_{ij})_{n \times n}$ is consistent, we take the logarithm on Eq. (14) and obtain

$$
\begin{cases}
(n-2) \log \left(\overline{q}_{1,ij}\right) = \oplus_{k=1, k \neq i, j}^{n} \left(\log \left(\overline{q}_{1,ik}\right) \oplus \log \left(\overline{q}_{1,kj}\right)\right) \\
(n-2) \log \left(\overline{q}_{2,ij}\right) = \oplus_{k=1, k \neq i, j}^{n} \left(\log \left(\overline{q}_{2,ik}\right) \oplus \log \left(\overline{q}_{2,kj}\right)\right)
\end{cases}
\tag{17}
$$

Eq. (17) indicates that

$$
\begin{cases}
(n-2) \log \left((\overline{b}_{1,ij})^{\theta_{ij}} \otimes (\overline{b}_{1,ij}^{\circ})^{1-\theta_{ij}}\right) = \oplus_{k=1, k \neq i, j}^{n} \left(\log \left((\overline{b}_{1,ik})^{\theta_{ik}}\right.\right. \\
\quad \left.\otimes (\overline{b}_{1,ik}^{\circ})^{1-\theta_{ik}}\right) \oplus \log \left((\overline{b}_{1,kj})^{\theta_{kj}} \otimes (\overline{b}_{1,kj}^{\circ})^{1-\theta_{kj}}\right)\right) \\
(n-2) \log \left((\overline{b}_{2,ij})^{\vartheta_{ij}} \otimes (\overline{b}_{2,ij}^{\circ})^{1-\vartheta_{ij}}\right) = \oplus_{k=1, k \neq i, j}^{n} \left(\log \left((\overline{b}_{2,ik})^{\vartheta_{ik}}\right.\right. \\
\quad \left.\otimes (\overline{b}_{2,ik}^{\circ})^{1-\vartheta_{ik}}\right) \oplus \log \left((\overline{b}_{2,kj})^{\vartheta_{kj}} \otimes (\overline{b}_{2,kj}^{\circ})^{1-\vartheta_{kj}}\right)\right)
\end{cases}
\tag{18}
$$

From Eq. (18), we derive

$$
\begin{cases}
(n-2) \left(\theta_{ij} \log(\rho_{ij}^-) + (1-\theta_{ij}) \log(\rho_{ij}^+)\right) \\
= \sum_{k=1, k \neq i, j}^{n} \left(\theta_{ik} \log(\rho_{ik}^-) + (1-\theta_{ik}) \log(\rho_{ik}^+) + \theta_{kj} \log(\rho_{kj}^-)\right. \\
\quad \left. +(1-\theta_{kj}) \log(\rho_{kj}^+)\right) \\
(n-2) \left(\theta_{ij} \log(\rho_{ij}^+) + (1-\theta_{ij}) \log(\rho_{ij}^-)\right) \\
= \sum_{k=1, k \neq i, j}^{n} \left(\theta_{ik} \log(\rho_{ik}^+) + (1-\theta_{ik}) \log(\rho_{ik}^-) + \theta_{kj} \log(\rho_{kj}^+)\right. \\
\quad \left. +(1-\theta_{kj}) \log(\rho_{kj}^-)\right) \\
(n-2) \left(\vartheta_{ij} \log(\sigma_{ij}^-) + (1-\vartheta_{ij}) \log(\sigma_{ij}^+)\right) \\
= \sum_{k=1, k \neq i, j}^{n} \left(\vartheta_{ik} \log(\sigma_{ik}^-) + (1-\vartheta_{ik}) \log(\sigma_{ik}^+) + \vartheta_{kj} \log(\sigma_{kj}^-)\right. \\
\quad \left. +(1-\vartheta_{kj}) \log(\sigma_{kj}^+)\right) \\
(n-2) \left(\vartheta_{ij} \log(\sigma_{ij}^+) + (1-\vartheta_{ij}) \log(\sigma_{ij}^-)\right) \\
= \sum_{k=1, k \neq i, j}^{n} \left(\vartheta_{ik} \log(\sigma_{ik}^+) + (1-\vartheta_{ik}) \log(\sigma_{ik}^-) + \vartheta_{kj} \log(\sigma_{kj}^+)\right. \\
\quad \left. +(1-\vartheta_{kj}) \log(\sigma_{kj}^-)\right)
\end{cases}
\tag{19}
$$

Eq. (19) shows that

$$
\begin{cases}
(n-2) \left(\theta_{ij} \log(\rho_{ij}^-) + (1-\theta_{ij}) \log(\rho_{ij}^+)\right) = \left(\sum_{k=1}^{i-1} + \sum_{k=i+1}^{j-1} + \sum_{k=j+1}^{n}\right) \\
\times \left(\theta_{ik} \log(\rho_{ik}^+) + (1-\theta_{ik}) \log(\rho_{ik}^-) + \theta_{kj} \log(\rho_{kj}^-) + (1-\theta_{kj}) \log(\rho_{kj}^+)\right) \\
(n-2) \left(\theta_{ij} \log(\rho_{ij}^+) + (1-\theta_{ij}) \log(\rho_{ij}^-)\right) = \left(\sum_{k=1}^{i-1} + \sum_{k=i+1}^{j-1} + \sum_{k=j+1}^{n}\right) \\
\times \left(\theta_{ik} \log(\rho_{ik}^+) + (1-\theta_{ik}) \log(\rho_{ik}^-) + \theta_{kj} \log(\rho_{kj}^+) + (1-\theta_{kj}) \log(\rho_{kj}^-)\right) \\
(n-2) \left(\vartheta_{ij} \log(\sigma_{ij}^-) + (1-\vartheta_{ij}) \log(\sigma_{ij}^+)\right) = \left(\sum_{k=1}^{i-1} + \sum_{k=i+1}^{j-1} + \sum_{k=j+1}^{n}\right) \\
\times \left(\vartheta_{ik} \log(\sigma_{ik}^-) + (1-\vartheta_{ik}) \log(\sigma_{ik}^+) + \vartheta_{kj} \log(\sigma_{kj}^-) + (1-\vartheta_{kj}) \log(\sigma_{kj}^+)\right) \\
(n-2) \left(\vartheta_{ij} \log(\sigma_{ij}^+) + (1-\vartheta_{ij}) \log(\sigma_{ij}^-)\right) = \left(\sum_{k=1}^{i-1} + \sum_{k=i+1}^{j-1} + \sum_{k=j+1}^{n}\right) \\
\times \left(\vartheta_{ik} \log(\sigma_{ik}^+) + (1-\vartheta_{ik}) \log(\sigma_{ik}^-) + \vartheta_{kj} \log(\sigma_{kj}^+) + (1-\vartheta_{kj}) \log(\sigma_{kj}^-)\right)
\end{cases}
\tag{20}
$$

Following the construction of elements in the IVMPRs $\overline{B}_1 = (\overline{b}_{1,ij})_{n \times n}$ and $\overline{B}_2 = (\overline{b}_{2,ij})_{n \times n}$, we can only apply the upper triangular part of

$\overline{B}_1$ and the lower triangular part of $\overline{B}_2$ to get unknown values in the incomplete IVIMPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$. From Eq. (20), we get

$$
\begin{cases}
\begin{aligned}
& \sum_{k=1}^{i-1}\left((\theta_{ki}-1)\log(\rho_{ki}^-)+\theta_{ki}\log(\rho_{ki}^+)+\theta_{kj}\log(\rho_{kj}^-)+(1-\theta_{kj})\log(\rho_{kj}^+)\right)\\
& +\sum_{k=i+1}^{j-1}\left(\theta_{ik}\log(\rho_{ik}^-)+(1-\theta_{ik})\log(\rho_{ik}^+)+\theta_{kj}\log(\rho_{kj}^-)+(1-\theta_{kj})\log(\rho_{kj}^+)\right)\\
& +\sum_{k=j+1}^{n}\left(\theta_{ik}\log(\rho_{ik}^-)+(1-\theta_{ik})\log(\rho_{ik}^+)+(\theta_{jk}-1)\log(\rho_{jk}^-)+\theta_{jk}\log(\rho_{jk}^+)\right)\\
& =(n-2)\left(\theta_{ij}\log(\rho_{ij}^-)+(1-\theta_{ij})\log(\rho_{ij}^+)\right),i,j=1,2,\ldots,n,i<j\\[4pt]
& \sum_{k=1}^{i-1}\left((\theta_{ki}-1)\log(\rho_{ki}^+)+\theta_{ki}\log(\rho_{ki}^-)+\theta_{kj}\log(\rho_{kj}^+)+(1-\theta_{kj})\log(\rho_{kj}^-)\right)\\
& +\sum_{k=i+1}^{j-1}\left(\theta_{ik}\log(\rho_{ik}^+)+(1-\theta_{ik})\log(\rho_{ik}^-)+\theta_{kj}\log(\rho_{kj}^+)+(1-\theta_{kj})\log(\rho_{kj}^-)\right)\\
& +\sum_{k=j+1}^{n}\left(\theta_{ik}\log(\rho_{ik}^+)+(1-\theta_{ik})\log(\rho_{ik}^-)+(\theta_{jk}-1)\log(\rho_{jk}^+)+\theta_{jk}\log(\rho_{jk}^-)\right)\\
& =(n-2)\left(\theta_{ij}\log(\rho_{ij}^+)+(1-\theta_{ij})\log(\rho_{ij}^-)\right),i,j=1,2,\ldots,n,i<j\\[4pt]
& \sum_{k=1}^{i-1}\left(\vartheta_{ik}\log(\sigma_{ik}^-)+(1-\vartheta_{ik})\log(\sigma_{ik}^+)+(\vartheta_{jk}-1)\log(\sigma_{jk}^-)+\vartheta_{jk}\log(\sigma_{jk}^+)\right)\\
& +\sum_{k=i+1}^{j-1}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^-)+\vartheta_{ki}\log(\sigma_{ki}^+)+(\vartheta_{jk}-1)\log(\sigma_{jk}^-)\right.\\
& \quad\left.+\vartheta_{jk}\log(\sigma_{jk}^+)\right)\\
& +\sum_{k=j+1}^{n}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^-)+\vartheta_{ki}\log(\sigma_{ki}^+)+\vartheta_{kj}\log(\sigma_{kj}^-)\right.\\
& \quad\left.+(1-\vartheta_{kj})\log(\sigma_{kj}^+)\right)\\
& =(n-2)\left(\vartheta_{ij}\log(\sigma_{ij}^-)+(1-\vartheta_{ij})\log(\sigma_{ij}^+)\right),i,j=1,2,\ldots,n,i>j\\[4pt]
& \sum_{k=1}^{i-1}\left(\vartheta_{ik}\log(\sigma_{ik}^+)+(1-\vartheta_{ik})\log(\sigma_{ik}^-)+(\vartheta_{jk}-1)\log(\sigma_{jk}^+)+\vartheta_{jk}\log(\sigma_{jk}^-)\right)\\
& +\sum_{k=i+1}^{j-1}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^+)+\vartheta_{ki}\log(\sigma_{ki}^-)+(\vartheta_{jk}-1)\log(\sigma_{jk}^+)\right.\\
& \quad\left.+\vartheta_{jk}\log(\sigma_{jk}^-)\right)\\
& +\sum_{k=j+1}^{n}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^+)+\vartheta_{ki}\log(\sigma_{ki}^-)+\vartheta_{kj}\log(\sigma_{kj}^+)\right.\\
& \quad\left.+(1-\vartheta_{kj})\log(\sigma_{kj}^-)\right)\\
& =(n-2)\left(\vartheta_{ij}\log(\sigma_{ij}^+)+(1-\vartheta_{ij})\log(\sigma_{ij}^-)\right),i,j=1,2,\ldots,n,i>j
\end{aligned}
\end{cases}
\tag{21}
$$

It is noted that Eqs. (17) to (21) are equivalent. The purpose for deriving Eq. (21) is to apply the upper triangular part of the QIVMPR $\overline{Q}_1 = (\overline{q}_{1,ij})_{n\times n}$ and the lower triangular part of the QIVMPR $\overline{Q}_2 = (\overline{q}_{2,ij})_{n\times n}$ to determine missing values.

To judge whether the incomplete IVIMPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ is consistent and to derive missing values, we establish the following model:

$$
\phi^* = \min\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\chi_{ij}^+ + \chi_{ij}^- + \gamma_{ij}^+ + \gamma_{ij}^-\right)\right.
$$

$$
\left.+ \sum_{i=j+1}^{n}\sum_{j=1}^{n-1}\left(\eta_{ij}^+ + \eta_{ij}^- + \mu_{ij}^+ + \mu_{ij}^-\right)\right)
$$

$$
s.t.
\begin{cases}
\begin{aligned}
& \sum_{k=1}^{i-1}\left((\theta_{ki}-1)\log(\rho_{ki}^-)+\theta_{ki}\log(\rho_{ki}^+)+\theta_{kj}\log(\rho_{kj}^-)+(1-\theta_{kj})\log(\rho_{kj}^+)\right)\\
& +\sum_{k=i+1}^{j-1}\left(\theta_{ik}\log(\rho_{ik}^-)+(1-\theta_{ik})\log(\rho_{ik}^+)+\theta_{kj}\log(\rho_{kj}^-)+(1-\theta_{kj})\log(\rho_{kj}^+)\right)\\
& +\sum_{k=j+1}^{n}\left(\theta_{ik}\log(\rho_{ik}^-)+(1-\theta_{ik})\log(\rho_{ik}^+)+(\theta_{jk}-1)\log(\rho_{jk}^-)+\theta_{jk}\log(\rho_{jk}^+)\right)\\
& =(n-2)\left(\theta_{ij}\log(\rho_{ij}^-)+(1-\theta_{ij})\log(\rho_{ij}^+)\right)-\chi_{ij}^+ + \chi_{ij}^-,i,j=1,2,\ldots,n,i<j\\[4pt]
& \sum_{k=1}^{i-1}\left((\theta_{ki}-1)\log(\rho_{ki}^+)+\theta_{ki}\log(\rho_{ki}^-)+\theta_{kj}\log(\rho_{kj}^+)+(1-\theta_{kj})\log(\rho_{kj}^-)\right)\\
& +\sum_{k=i+1}^{j-1}\left(\theta_{ik}\log(\rho_{ik}^+)+(1-\theta_{ik})\log(\rho_{ik}^-)+\theta_{kj}\log(\rho_{kj}^+)+(1-\theta_{kj})\log(\rho_{kj}^-)\right)\\
& +\sum_{k=j+1}^{n}\left(\theta_{ik}\log(\rho_{ik}^+)+(1-\theta_{ik})\log(\rho_{ik}^-)+(\theta_{jk}-1)\log(\rho_{jk}^+)+\theta_{jk}\log(\rho_{jk}^-)\right)\\
& =(n-2)\left(\theta_{ij}\log(\rho_{ij}^+)+(1-\theta_{ij})\log(\rho_{ij}^-)\right)-\gamma_{ij}^+ + \gamma_{ij}^-,i,j=1,2,\ldots,n,i<j\\[4pt]
& \sum_{k=1}^{i-1}\left(\vartheta_{ik}\log(\sigma_{ik}^-)+(1-\vartheta_{ik})\log(\sigma_{ik}^+)+(\vartheta_{jk}-1)\log(\sigma_{jk}^-)+\vartheta_{jk}\log(\sigma_{jk}^+)\right)\\
& +\sum_{k=i+1}^{j-1}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^-)+\vartheta_{ki}\log(\sigma_{ki}^+)+(\vartheta_{jk}-1)\log(\sigma_{jk}^-)+\vartheta_{jk}\log(\sigma_{jk}^+)\right)\\
& +\sum_{k=j+1}^{n}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^-)+\vartheta_{ki}\log(\sigma_{ki}^+)+\vartheta_{kj}\log(\sigma_{kj}^-)+(1-\vartheta_{kj})\log(\sigma_{kj}^+)\right)\\
& =(n-2)\left(\vartheta_{ij}\log(\sigma_{ij}^-)+(1-\vartheta_{ij})\log(\sigma_{ij}^+)\right)-\eta_{ij}^+ + \eta_{ij}^-,i,j=1,2,\ldots,n,i>j\\[4pt]
& \sum_{k=1}^{i-1}\left(\vartheta_{ik}\log(\sigma_{ik}^+)+(1-\vartheta_{ik})\log(\sigma_{ik}^-)+(\vartheta_{jk}-1)\log(\sigma_{jk}^+)+\vartheta_{jk}\log(\sigma_{jk}^-)\right)\\
& +\sum_{k=i+1}^{j-1}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^+)+\vartheta_{ki}\log(\sigma_{ki}^-)+(\vartheta_{jk}-1)\log(\sigma_{jk}^+)+\vartheta_{jk}\log(\sigma_{jk}^-)\right)\\
& +\sum_{k=j+1}^{n}\left((\vartheta_{ki}-1)\log(\sigma_{ki}^+)+\vartheta_{ki}\log(\sigma_{ki}^-)+\vartheta_{kj}\log(\sigma_{kj}^+)+(1-\vartheta_{kj})\log(\sigma_{kj}^-)\right)\\
& =(n-2)\left(\vartheta_{ij}\log(\sigma_{ij}^+)+(1-\vartheta_{ij})\log(\sigma_{ij}^-)\right)-\mu_{ij}^+ + \mu_{ij}^-,i,j=1,2,\ldots,n,i>j\\[4pt]
& 1/9 \le \rho_{ij}^- \le \rho_{ij}^+, \rho_{ij}^- \in U_1 \wedge \rho_{ij}^+ \notin U_2\\
& \rho_{ij}^- \le \rho_{ij}^+ \le \min\{1/\sigma_{ij}^-, 9\}, \rho_{ij}^- \notin U_1 \wedge \rho_{ij}^+ \in U_2 \wedge \sigma_{ij}^- \notin U_4\\
& 1/9 \le \sigma_{ij}^- \le \sigma_{ij}^+, \sigma_{ij}^- \in U_3 \wedge \sigma_{ij}^+ \notin U_4\\
& \sigma_{ij}^- \le \sigma_{ij}^+ \le \min\{1/\rho_{ij}^-, 9\}, \sigma_{ij}^- \notin U_3 \wedge \rho_{ij}^+ \in U_4 \wedge \rho_{ij}^- \notin U_2\\
& cvskip[3pt]\ 1/9 \le \rho_{ij}^- \le \rho_{ij}^+ \le \min\{1/\sigma_{ij}^-, 9\}, \rho_{ij}^- \in U_1 \wedge \rho_{ij}^+ \in U_2 \wedge \sigma_{ij}^- \notin U_4\\
& 1/9 \le \sigma_{ij}^- \le \sigma_{ij}^+ \le \min\{1/\rho_{ij}^-, 9\}, \sigma_{ij}^- \in U_3 \wedge \sigma_{ij}^+ \in U_4 \wedge \rho_{ij}^- \notin U_2\\
& \rho_{ij}^- \le \rho_{ij}^+ \le 9, \sigma_{ij}^- \le \sigma_{ij}^+ \le 9,
\begin{cases}\log(\rho_{ij}^+)+\log(\sigma_{ij}^-)\le 0\\ \log(\rho_{ij}^-)+\log(\sigma_{ij}^+)\le 0\end{cases} \text{ for }\\
& \begin{cases}\rho_{ij}^- \notin U_1 \wedge \rho_{ij}^+ \in U_2\\ \sigma_{ij}^- \notin U_3 \wedge \sigma_{ij}^+ \in U_4\end{cases}\\
& \begin{cases}1/9 \le \rho_{ij}^- \le \rho_{ij}^+ \le 9\\ 1/9 \le \sigma_{ij}^- \le \sigma_{ij}^+ \le 9\end{cases}, \begin{cases}\log(\rho_{ij}^+)+\log(\sigma_{ij}^-)\le 0\\ \log(\rho_{ij}^-)+\log(\sigma_{ij}^+)\le 0\end{cases}, \begin{cases}\rho_{ij}^- \in U_1 \wedge \rho_{ij}^+ \in U_2\\ \sigma_{ij}^- \in U_3 \wedge \sigma_{ij}^+ \in U_4\end{cases}\\
& \chi_{ij}^+, \chi_{ij}^-, \gamma_{ij}^+, \gamma_{ij}^- \ge 0, i,j=1,2,\ldots,n; i<j\\
& \theta_{ij}=0 \vee 1, i,j=1,2,\ldots,n; i<j\\
& \eta_{ij}^+, \eta_{ij}^-, \mu_{ij}^+, \mu_{ij}^- \ge 0, i,j=1,2,\ldots,n; i>j\\
& \vartheta_{ij}=0 \vee 1, i,j=1,2,\ldots,n; i>j
\end{aligned}
\end{cases}
\tag{M-4}
$$

where the first four constraints are derived from Eq. (21) by adding non-negative flexible variables $\chi_{k,ij}^+$, $\chi_{k,ij}^-$, $\gamma_{k,ij}^+$, $\gamma_{k,ij}^-$, $\eta_{k,ij}^+$, $\eta_{k,ij}^-$, $\mu_{k,ij}^+$, and

$\mu_{k,ij}^-$ for each triple of (i, k, j) such that $i < k < j$, θ_{ij} and ϑ_{ij} are 0–1 indicator variables as shown in Eq. (9), the rest constraints are derived from the conditions of IVIMVs in IVIMPRs as shown in Eq. (5), and U_p are shown above, $p = 1, 2, 3, 4$.

Addressing model (M-4), we derive missing values in $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$. If we have $\phi^* = 0$, then $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ is consistent. Otherwise, it is inconsistent. However, the determined missing values make $\tilde{R}$ have the highest consistent level.

4.2. Models to derive consistent IVIMPRs

In general, preference relations offered by the DMs are inconsistent (Chiclana et al., 2009). To derive the reasonable ranking, consistency analysis is needed. Considering the consistency of IVIMPRs, this subsection builds several models to derive completely consistent IVIMPRs from inconsistent ones.

Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be a complete IVIMPR. To judge its consistency, we can apply model (M-3). However, when the objective function value $\varphi^* \neq 0$, $\tilde{R}$ is inconsistent. In this case, we should adjust the original judgements. To do this, we build the following model to determine QIVMPRs with the highest consistent level.

$$\Delta^* = \max \left(\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \theta_{ij} \right) - \left(\sum_{i=j+1}^{n} \sum_{j=1}^{n-1} \vartheta_{ij} \right) \right)$$

$$
s.t. \begin{cases}
\varphi^* = \left(\sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \left(\chi_{k,ij}^+ + \chi_{k,ij}^- + \gamma_{k,ij}^+ + \gamma_{k,ij}^- \right) \right. \\
\left. + \sum_{i=k+1}^{n} \sum_{k=j+1}^{n-1} \sum_{j=1}^{n-2} \left(\chi_{k,ij}^+ + \chi_{k,ij}^- + \gamma_{k,ij}^+ + \gamma_{k,ij}^- \right) \right) \\
\theta_{ik} \log(\rho_{ik}^-) + (1 - \theta_{ik}) \log(\rho_{ik}^+) + \theta_{kj} \log(\rho_{kj}^-) + (1 - \theta_{kj}) \log(\rho_{kj}^+) \\
= \theta_{ij} \log(\rho_{ij}^-) + (1 - \theta_{ij}) \log(\rho_{ij}^+) - \chi_{k,ij}^+ + \chi_{k,ij}^-, \\
\quad i, k, j = 1, 2, \ldots, n; i < k < j \\
\theta_{ik} \log(\rho_{ik}^+) + (1 - \theta_{ik}) \log(\rho_{ik}^-) + \theta_{kj} \log(\rho_{kj}^+) + (1 - \theta_{kj}) \log(\rho_{kj}^-) \\
= \theta_{ij} \log(\rho_{ij}^+) + (1 - \theta_{ij}) \log(\rho_{ij}^-) - \gamma_{k,ij}^+ + \gamma_{k,ij}^-, \\
\quad i, k, j = 1, 2, \ldots, n; i < k < j \\
\chi_{k,ij}^+, \chi_{k,ij}^-, \gamma_{k,ij}^+, \gamma_{k,ij}^- \geq 0, i, k, j = 1, 2, \ldots, n; i < k < j \\
\theta_{ij} = 0 \vee 1, i, j = 1, 2, \ldots, n; i < j \\
\vartheta_{ik} \log(\sigma_{ik}^-) + (1 - \vartheta_{ik}) \log(\sigma_{ik}^+) + \vartheta_{kj} \log(\sigma_{kj}^-) + (1 - \vartheta_{kj}) \log(\sigma_{kj}^+) \\
= \vartheta_{ij} \log(\sigma_{ij}^-) + (1 - \vartheta_{ij}) \log(\sigma_{ij}^+) - \eta_{k,ij}^+ + \eta_{k,ij}^-, \\
\quad i, k, j = 1, 2, \ldots, n; i > k > j \\
\vartheta_{ik} \log(\sigma_{ik}^+) + (1 - \vartheta_{ik}) \log(\sigma_{ik}^-) + \vartheta_{kj} \log(\sigma_{kj}^+) + (1 - \vartheta_{kj}) \log(\sigma_{kj}^-) \\
= \vartheta_{ij} \log(\sigma_{ij}^+) + (1 - \vartheta_{ij}) \log(\sigma_{ij}^-) - \mu_{k,ij}^+ + \mu_{k,ij}^-, \\
\quad i, k, j = 1, 2, \ldots, n; i > k > j \\
\eta_{k,ij}^+, \eta_{k,ij}^-, \mu_{k,ij}^+, \mu_{k,ij}^- \geq 0, i, k, j = 1, 2, \ldots, n; i > k > j \\
\vartheta_{ij} = 0 \vee 1, i, j = 1, 2, \ldots, n; i > j
\end{cases}
$$

(M-5)

where φ^* is the optimal objective function value of model (M-3), and all other constraints are same as those in model (M-3).

Addressing model (M-5), we derive two associated QIVMPRs $\overline{Q}'_1 = (\overline{q}'_{1,ij})_{n \times n}$ and $\overline{Q}'_2 = (\overline{q}'_{2,ij})_{n \times n}$ by the obtained 0–1 IVs. Furthermore, the upper triangular part of the QIVMPR $\overline{Q}'_1$ has the largest number of intervals, while the lower triangular part of the QIVMPR $\overline{Q}'_2$ has the largest number of intervals.

For the QIVMPR $\overline{Q}'_1 = (\overline{q}'_{1,ij})_{n \times n}$, let $\overline{Q}^*_1 = (\overline{q}^*_{1,ij})_{n \times n}$ be the adjusted consistent QIVMPR, where

$$\overline{q}^*_{1,ij} = \overline{q}'_{1,ij} \otimes [\varsigma_{1,ij}^-, \varsigma_{1,ij}^+] \otimes [1/\tau_{1,ij}^-, 1/\tau_{1,ij}^+] \tag{22}$$

with $\varsigma_{1,ij}^-, \varsigma_{1,ij}^+, \tau_{1,ij}^-, \tau_{1,ij}^+ \geq 1$ and $\varsigma_{1,ij}^- \varsigma_{1,ji}^- = \tau_{1,ij}^- \tau_{1,ji}^- = \varsigma_{1,ij}^+ \varsigma_{1,ji}^+ = \tau_{1,ij}^+ \tau_{1,ji}^+ = 1$, $i, j = 1, 2, \ldots, n$.

For the QIVMPR $\overline{Q}'_2 = (\overline{q}'_{2,ij})_{n \times n}$, let $\overline{Q}^*_2 = (\overline{q}^*_{2,ij})_{n \times n}$ be the adjusted consistent QIVMPR, where

$$\overline{q}^*_{2,ij} = \overline{q}'_{2,ij} \otimes [\varsigma_{2,ij}^-, \varsigma_{2,ij}^+] \otimes [1/\tau_{2,ij}^-, 1/\tau_{2,ij}^+] \tag{23}$$

with $\varsigma_{2,ij}^-, \varsigma_{2,ij}^+, \tau_{2,ij}^-, \tau_{2,ij}^+ \geq 1$ and $\varsigma_{2,ij}^- \varsigma_{2,ji}^- = \tau_{2,ij}^- \tau_{2,ji}^- = \varsigma_{2,ij}^+ \varsigma_{2,ji}^+ = \tau_{2,ij}^+ \tau_{2,ji}^+ = 1$, $i, j = 1, 2, \ldots, n$.

From Eqs. (22) and (23), we derive

$$
\begin{cases}
\left[\dfrac{q_{1,ij}'^- \varsigma_{1,ij}^-}{\tau_{1,ij}^-}, \dfrac{q_{1,ij}'^+ \varsigma_{1,ij}^+}{\tau_{1,ij}^+} \right] = \left[\dfrac{q_{1,ik}'^- \varsigma_{1,ik}^-}{\tau_{1,ik}^-}, \dfrac{q_{1,ik}'^+ \varsigma_{1,ik}^+}{\tau_{1,ik}^+} \right] \\
\otimes \left[\dfrac{q_{1,kj}'^- \varsigma_{1,kj}^-}{\tau_{1,kj}^-}, \dfrac{q_{1,kj}'^+ \varsigma_{1,kj}^+}{\tau_{1,kj}^+} \right], i, k, j = 1, 2, \ldots, n, i < k < j \\
\left[\dfrac{q_{2,ij}'^- \varsigma_{2,ij}^-}{\tau_{2,ij}^-}, \dfrac{q_{2,ij}'^+ \varsigma_{2,ij}^+}{\tau_{2,ij}^+} \right] = \left[\dfrac{q_{2,ik}'^- \varsigma_{2,ik}^-}{\tau_{2,ik}^-}, \dfrac{q_{2,ik}'^+ \varsigma_{2,ik}^+}{\tau_{2,ik}^+} \right] \\
\otimes \left[\dfrac{q_{2,kj}'^- \varsigma_{2,kj}^-}{\tau_{2,kj}^-}, \dfrac{q_{2,kj}'^+ \varsigma_{2,kj}^+}{\tau_{2,kj}^+} \right], i, k, j = 1, 2, \ldots, n, i > k > j
\end{cases}
\tag{24}
$$

We take the logarithm on Eq. (24) and derive

$$
\begin{cases}
\log(q_{1,ij}'^-) + \log(\varsigma_{1,ij}^-) - \log(\tau_{1,ij}^-) = \log(q_{1,ik}'^-) \\
+ \log(\varsigma_{1,ik}^-) \\
- \log(\tau_{1,ik}^-) + \log(q_{1,kj}'^-) + \log(\varsigma_{1,kj}^-) - \log(\tau_{1,kj}^-) \\
\log(q_{1,ij}'^+) + \log(\varsigma_{1,ij}^+) - \log(\tau_{1,ij}^+) = \log(q_{1,ik}'^+) \\
+ \log(\varsigma_{1,ik}^+) \\
- \log(\tau_{1,ik}^+) + \log(q_{1,kj}'^+) + \log(\varsigma_{1,kj}^+) - \log(\tau_{1,kj}^+) \\
\log(q_{2,ij}'^-) + \log(\varsigma_{2,ij}^-) - \log(\tau_{2,ij}^-) = \log(q_{2,ik}'^-) \\
+ \log(\varsigma_{2,ik}^-) \\
- \log(\tau_{2,ik}^-) + \log(q_{2,kj}'^-) + \log(\varsigma_{2,kj}^-) - \log(\tau_{2,kj}^-) \\
\log(q_{2,ij}'^+) + \log(\varsigma_{2,ij}^+) - \log(\tau_{2,ij}^+) = \log(q_{2,ik}'^+) \\
+ \log(\varsigma_{2,ik}^+) \\
- \log(\tau_{2,ik}^+) + \log(q_{2,kj}'^+) + \log(\varsigma_{2,kj}^+) - \log(\tau_{2,kj}^+)
\end{cases}
\begin{matrix}
i, k, j = 1, 2, \ldots, n \\
i < k < j \\
\\
\\
i, k, j = 1, 2, \ldots, n \\
i > k > j
\end{matrix}
\tag{25}
$$

The adjustments should be as small as possible to retain more original information. Thus, we further build the following model:

$$\Lambda^* = \min \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \Big(\varsigma_{1,ij}^- + \varsigma_{1,ij}^+ + \tau_{1,ij}^- + \tau_{1,ij}^+ + \varsigma_{2,ji}^-$$
$$+ \varsigma_{2,ji}^+ + \tau_{2,ji}^- + \tau_{2,ji}^+ \Big)$$

$$s.t. \begin{cases} \begin{cases} \log(q'^-_{1,ij}) + \log(\varsigma^-_{1,ij}) - \log(\tau^-_{1,ij}) = \log(q'^-_{1,ik}) \\ + \log(\varsigma^-_{1,ik}) \\ - \log(\tau^-_{1,ik}) + \log(q'^-_{1,kj}) + \log(\varsigma^-_{1,kj}) - \log(\tau^-_{1,kj}) \\ \log(q'^+_{1,ij}) + \log(\varsigma^+_{1,ij}) - \log(\tau^+_{1,ij}) = \log(q'^+_{1,ik}) \\ + \log(\varsigma^+_{1,ik}) \\ - \log(\tau^+_{1,ik}) + \log(q'^+_{1,kj}) + \log(\varsigma^+_{1,kj}) - \log(\tau^+_{1,kj}) \end{cases} \begin{matrix} i,k,j = 1,2,\dots,n \\ i < k < j \end{matrix} \\[2em] \begin{cases} \log(q'^-_{2,ij}) + \log(\varsigma^-_{2,ij}) - \log(\tau^-_{2,ij}) = \log(q'^-_{2,ik}) \\ + \log(\varsigma^-_{2,ik}) \\ - \log(\tau^-_{2,ik}) + \log(q'^-_{2,kj}) + \log(\varsigma^-_{2,kj}) - \log(\tau^-_{2,kj}) \\ \log(q'^+_{2,ij}) + \log(\varsigma^+_{2,ij}) - \log(\tau^+_{2,ij}) = \log(q'^+_{2,ik}) \\ + \log(\varsigma^+_{2,ik}) \\ - \log(\tau^+_{2,ik}) + \log(q'^+_{2,kj}) + \log(\varsigma^+_{2,kj}) - \log(\tau^+_{2,kj}) \end{cases} \begin{matrix} i,k,j = 1,2,\dots,n \\ i > k > j \end{matrix} \\[2em] \begin{cases} \log(q'^-_{1,ij}) + \log(\varsigma^-_{1,ij}) - \log(\tau^-_{1,ij}) \le \log(9) \\ \log(q'^-_{1,ij}) + \log(\varsigma^-_{1,ij}) - \log(\tau^-_{1,ij}) \ge -\log(9) \\ \log(q'^+_{1,ij}) + \log(\varsigma^+_{1,ij}) - \log(\tau^+_{1,ij}) \le \log(9) \\ \log(q'^+_{1,ij}) + \log(\varsigma^+_{1,ij}) - \log(\tau^+_{1,ij}) \ge -\log(9) \end{cases} \begin{matrix} i,k,j = 1,2,\dots,n \\ i < k < j \end{matrix} \\[2em] \begin{cases} \log(q'^-_{2,ij}) + \log(\varsigma^-_{2,ij}) - \log(\tau^-_{2,ij}) \le \log(9) \\ \log(q'^-_{2,ij}) + \log(\varsigma^-_{2,ij}) - \log(\tau^-_{2,ij}) \ge -\log(9) \\ \log(q'^+_{2,ij}) + \log(\varsigma^+_{2,ij}) - \log(\tau^+_{2,ij}) \le \log(9) \\ \log(q'^+_{2,ij}) + \log(\varsigma^+_{2,ij}) - \log(\tau^+_{2,ij}) \ge -\log(9) \end{cases} \begin{matrix} i,k,j = 1,2,\dots,n \\ i > k > j \end{matrix} \\[2em] \varsigma^-_{1,ij}, \varsigma^+_{1,ij}, \tau^-_{1,ij}, \tau^+_{1,ij} > 1, i,k,j = 1,2,\dots,n, i < k < j \\ \varsigma^-_{2,ij}, \varsigma^+_{2,ij}, \tau^-_{2,ij}, \tau^+_{2,ij} > 1, i,k,j = 1,2,\dots,n, i > k > j \end{cases}$$

(M-6)

where the first four constraints are derived from Eq. (25), $\varsigma^-_{1,ij}, \varsigma^+_{1,ij}$, $\tau^-_{1,ij}, \tau^+_{1,ij}, \varsigma^-_{2,ij}, \varsigma^+_{2,ij}, \tau^-_{2,ij}$ and $\tau^+_{2,ij}$ are adjusted variables as shown in Eqs. (22) and (23), and the fifth to twelfth constraints are obtained from the conditions of IVIMVs in IVIMPRs as shown in Eq. (5).

To reduce the complexity of obtaining consistent QIVMPRs, we convert model (M-6) into the following linear model:

$$\Omega^* = \min \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \Big(\xi_{1,ij}^- + \xi_{1,ij}^+ + \zeta_{1,ij}^- + \zeta_{1,ij}^+ + \xi_{2,ji}^-$$
$$+ \xi_{2,ji}^+ + \zeta_{2,ji}^- + \zeta_{2,ji}^+ \Big)$$

$$s.t. \begin{cases} \begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} = \log(q'^-_{1,ik}) + \xi^-_{1,ik} - \zeta^-_{1,ik} \\ + \log(q'^-_{1,kj}) + \xi^-_{1,kj} - \zeta^-_{1,kj} \\ \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} = \log(q'^+_{1,ik}) + \xi^+_{1,ik} - \zeta^+_{1,ik} \\ + \log(q'^+_{1,kj}) + \xi^+_{1,kj} - \zeta^+_{1,kj} \\ \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} = \log(q'^-_{2,ki}) + \xi^-_{2,ki} - \zeta^-_{2,ki} \\ + \log(q'^-_{2,jk}) + \xi^-_{2,jk} - \zeta^-_{2,jk} \\ \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji} = \log(q'^+_{2,ki}) + \xi^+_{2,ki} - \zeta^+_{2,ki} \\ + \log(q'^+_{2,jk}) + \xi^+_{2,jk} - \zeta^+_{2,jk} \end{cases} \begin{matrix} i,k,j = 1,2,\dots,n \\ i < k < j \end{matrix} \\[2em] \log(q'^-_{1,ij}) + \log(\varsigma^-_{1,ij}) - \log(\tau^-_{1,ij}) \le \log(9) \\ \log(q'^-_{1,ij}) + \log(\varsigma^-_{1,ij}) - \log(\tau^-_{1,ij}) \ge -\log(9) \\ \log(q'^+_{1,ij}) + \log(\varsigma^+_{1,ij}) - \log(\tau^+_{1,ij}) \le \log(9) \\ \log(q'^+_{1,ij}) + \log(\varsigma^+_{1,ij}) - \log(\tau^+_{1,ij}) \ge -\log(9) \\ \log(q'^-_{2,ji}) + \log(\varsigma^-_{2,ji}) - \log(\tau^-_{2,ji}) \le \log(9) \\ \log(q'^-_{2,ji}) + \log(\varsigma^-_{2,ji}) - \log(\tau^-_{2,ji}) \ge -\log(9) \\ \log(q'^+_{2,ji}) + \log(\varsigma^+_{2,ji}) - \log(\tau^+_{2,ji}) \le \log(9) \\ \log(q'^+_{2,ji}) + \log(\varsigma^+_{2,ji}) - \log(\tau^+_{2,ji}) \ge -\log(9) \\ \xi^-_{1,ij}, \zeta^-_{1,ij}, \xi^+_{1,ij}, \zeta^+_{1,ij}, \xi^-_{2,ji}, \zeta^-_{2,ji}, \xi^+_{2,ji}, \zeta^+_{2,ji} \ge 0, i,k,j = 1,2,\dots,n, \\ \quad i < k < j \end{cases}$$

(M-7)

where $\begin{cases} \xi^-_{1,ij} = \log(\varsigma^-_{1,ij}), \xi^+_{1,ij} = \log(\varsigma^+_{1,ij}) \\ \xi^-_{2,ji} = \log(\varsigma^-_{2,ji}), \xi^+_{2,ji} = \log(\varsigma^+_{2,ji}) \end{cases}$ and
$\begin{cases} \zeta^-_{1,ij} = \log(\tau^-_{1,ij}), \zeta^+_{1,ij} = \log(\tau^+_{1,ij}) \\ \zeta^-_{2,ji} = \log(\tau^-_{2,ji}), \zeta^+_{2,ji} = \log(\tau^+_{2,ji}), \end{cases} i,k,j = 1,2,\dots,n$ with $i < k < j$, and all other notations are same as those in model (M-6).

Remark 2. Model (M-7) cannot guarantee that adjusted consistent IVMPRs satisfy the condition of IVIFVs, namely, none of the following conditions holds:

From (i) $\begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} \le \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} \\ \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} > \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji} \end{cases}$ or

(ii) $\begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} > \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} \\ \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} \le \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji}, \end{cases}$

we have $\begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} + \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} \le 0 \\ \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} + \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji} \le 0. \end{cases}$

From (iii) $\begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} \le \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} \\ \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} \le \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji} \end{cases}$ or

(iv) $\begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} > \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} \\ \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} > \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji}, \end{cases}$

we derive $\begin{cases} \log(q'^-_{1,ij}) + \xi^-_{1,ij} - \zeta^-_{1,ij} + \log(q'^+_{2,ji}) + \xi^+_{2,ji} - \zeta^+_{2,ji} \le 0 \\ \log(q'^+_{1,ij}) + \xi^+_{1,ij} - \zeta^+_{1,ij} + \log(q'^-_{2,ji}) + \xi^-_{2,ji} - \zeta^-_{2,ji} \le 0. \end{cases}$

To avoid the above situation, we further build the following linear model:

$$\Phi^* = \min \sum_{i=1}^{n-2} \sum_{k=i+1}^{n-1} \sum_{j=k+1}^{n} \Big(\xi_{1,ij}^- + \xi_{1,ij}^+ + \zeta_{1,ij}^- + \zeta_{1,ij}^+ + \xi_{2,ji}^-$$
$$+ \xi_{2,ji}^+ + \zeta_{2,ji}^- + \zeta_{2,ji}^+ \Big)$$

$$s.t. \begin{cases} \begin{cases} \log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij} = \log(q'^{-}_{1,ik}) + \xi^{-}_{1,ik} - \zeta^{-}_{1,ik} \\ \quad + \log(q'^{-}_{1,kj}) + \xi^{-}_{1,kj} - \zeta^{-}_{1,kj} \\ \log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij} = \log(q'^{+}_{1,ik}) + \xi^{+}_{1,ik} - \zeta^{+}_{1,ik} \\ \quad + \log(q'^{+}_{1,kj}) + \xi^{+}_{1,kj} - \zeta^{+}_{1,kj} \\ \log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji} = \log(q'^{-}_{2,ki}) + \xi^{-}_{2,ki} - \zeta^{-}_{2,ki} \\ \quad + \log(q'^{-}_{2,jk}) + \xi^{-}_{2,jk} - \zeta^{-}_{2,jk} \\ \log(q'^{+}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji} = \log(q'^{+}_{2,ki}) + \xi^{+}_{2,ki} - \zeta^{+}_{2,ki} \\ \quad + \log(q'^{+}_{2,jk}) + \xi^{+}_{2,jk} - \zeta^{+}_{2,jk} \\ \log(q'^{-}_{1,ij}) + \log(\varsigma^{-}_{1,ij}) - \log(\tau^{-}_{1,ij}) \leq \log(9) \\ \log(q'^{-}_{1,ij}) + \log(\varsigma^{-}_{1,ij}) - \log(\tau^{-}_{1,ij}) \geq -\log(9) \\ \log(q'^{+}_{1,ij}) + \log(\varsigma^{+}_{1,ij}) - \log(\tau^{+}_{1,ij}) \leq \log(9) \\ \log(q'^{+}_{1,ij}) + \log(\varsigma^{+}_{1,ij}) - \log(\tau^{+}_{1,ij}) \geq -\log(9) \\ \log(q'^{-}_{2,ji}) + \log(\varsigma^{-}_{2,ji}) - \log(\tau^{-}_{2,ji}) \leq \log(9) \\ \log(q'^{-}_{2,ji}) + \log(\varsigma^{-}_{2,ji}) - \log(\tau^{-}_{2,ji}) \geq -\log(9) \\ \log(q'^{+}_{2,ji}) + \log(\varsigma^{+}_{2,ji}) - \log(\tau^{+}_{2,ji}) \leq \log(9) \\ \log(q'^{+}_{2,ji}) + \log(\varsigma^{+}_{2,ji}) - \log(\tau^{+}_{2,ji}) \geq -\log(9) \end{cases} \begin{matrix} i,k,j = 1,2,\dots,n \\ i < k < j \end{matrix} \\[2em]

\begin{cases} \theta^{1}_{ij}\left(\log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij}\right. \\ \quad \left. \leq \log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij}\right) \\ \theta^{1}_{ij}\left(\log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji}\right. \\ \quad \left. > \log(q'^{+}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji}\right) \\ \theta^{2}_{ij}\left(\log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij}\right. \\ \quad \left. > \log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij}\right) \\ \theta^{2}_{ij}\left(\log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji}\right. \\ \quad \left. \leq \log(q'^{+}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji}\right) \\ \left(\theta^{1}_{ij} + \theta^{2}_{ij}\right)\left(\log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij}\right. \\ \quad \left. + \log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji}\right) \leq 0 \\ \left(\theta^{1}_{ij} + \theta^{2}_{ij}\right)\left(\log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij}\right. \\ \quad \left. + \log(q'^{-}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji}\right) \leq 0 \end{cases} \begin{matrix} i,j = 1,2,\dots,n, i < j \end{matrix} \\[2em]

\begin{cases} \theta^{3}_{ij}\left(\log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij}\right. \\ \quad \left. \leq \log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij}\right) \\ \theta^{3}_{ij}\left(\log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji}\right. \\ \quad \left. \leq \log(q'^{+}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji}\right) \\ \theta^{4}_{ij}\left(\log(q'^{-}_{1,ij}) + \xi^{-}_{1,ij} - \zeta^{-}_{1,ij}\right. \\ \quad \left. > \log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij}\right) \\ \theta^{4}_{ij}\left(\log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji}\right. \\ \quad \left. > \log(q'^{+}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji}\right) \\ \left(\theta^{3}_{ij} + \theta^{4}_{ij}\right)\left(\log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij}\right. \\ \quad \left. + \log(q'^{-}_{2,ji}) + \xi^{-}_{2,ji} - \zeta^{-}_{2,ji}\right) \leq 0 \\ \left(\theta^{3}_{ij} + \theta^{4}_{ij}\right)\left(\log(q'^{+}_{1,ij}) + \xi^{+}_{1,ij} - \zeta^{+}_{1,ij}\right. \\ \quad \left. + \log(q'^{+}_{2,ji}) + \xi^{+}_{2,ji} - \zeta^{+}_{2,ji}\right) \leq 0 \end{cases} \begin{matrix} i,j = 1,2,\dots,n, i < j \end{matrix} \\[2em]

\theta^{1}_{ij} = 1 \vee 0, \theta^{2}_{ij} = 1 \vee 0, \theta^{3}_{ij} = 1 \vee 0, \theta^{4}_{ij} = 1 \vee 0, i,j = 1,2,\dots,n, i < j \\
\theta^{1}_{ij} + \theta^{2}_{ij} + \theta^{3}_{ij} + \theta^{4}_{ij} = 1, i,j = 1,2,\dots,n, i < j \\
\xi^{-}_{1,ij}, \varsigma^{-}_{1,ij}, \xi^{+}_{1,ij}, \varsigma^{+}_{1,ij}, \xi^{-}_{2,ji}, \varsigma^{-}_{2,ji}, \xi^{+}_{2,ji}, \varsigma^{+}_{2,ji} \geq 0, i,k,j = 1,2,\dots,n, \\
\quad i < k < j \end{cases} \tag{M-8}$$

where the first eleven constraints are obtained from model (M-7), θ^{p}_{ij} is an 0–1 indicator variable for all $i, j = 1, 2, \dots, n$ with $i < j$ and $p = 1, 2, 3, 4$, and the rest constraints are derived the four cases listed in Remark 2.

Addressing model (M-8), we derive the adjusted consistent QIVM-PRs $\overline{Q}^{*}_1 = (\overline{q}^{*}_{1,ij})_{n\times n}$ and $\overline{Q}^{*}_2 = (\overline{q}^{*}_{2,ij})_{n\times n}$ with the smallest total adjustment. Then, we can obtain the associated consistent IVMPRs $\overline{B}^{*}_1 = (\overline{b}^{*}_{1,ij})_{n\times n}$ and $\overline{B}^{*}_2 = (\overline{b}^{*}_{2,ij})_{n\times n}$, where $\overline{b}^{*}_{1,ij} = \begin{cases} \overline{q}^{*}_{1,ij} & q^{*,-}_{1,ij} \leq q^{*,+}_{1,ij} \\ \overline{q}^{*\circ}_{1,ij} & q^{*,-}_{1,ij} > q^{*,+}_{1,ij} \end{cases}$ and $\overline{b}^{*}_{2,ij} = \begin{cases} \overline{q}^{*}_{2,ij} & q^{*,-}_{2,ij} \leq q^{*,+}_{2,ij} \\ \overline{q}^{*\circ}_{2,ij} & q^{*,-}_{2,ij} > q^{*,+}_{2,ij} \end{cases}$, $i, j = 1, 2, \dots, n$ with $i < j$. Furthermore, the consistent IVIMPR $\tilde{R}^{*} = (\tilde{r}^{*}_{ij})_{n\times n}$ is derived as:

$$\tilde{r}^{*}_{ij} = \left(\overline{\rho}^{*}_{ij}, \sigma^{*}_{ij}\right) = \begin{cases} \left(\overline{q}^{*}_{1,ij}, \overline{q}^{*}_{2,ij}\right) & q^{*,-}_{1,ij} \leq q^{*,+}_{1,ij} \wedge q^{*,-}_{2,ij} \leq q^{*,+}_{2,ij} \\ \left(\overline{q}^{*\circ}_{1,ij}, \overline{q}^{*}_{2,ij}\right) & q^{*,-}_{1,ij} > q^{*,+}_{1,ij} \wedge q^{*,-}_{2,ij} \leq q^{*,+}_{2,ij} \\ \left(\overline{q}^{*}_{1,ij}, \overline{q}^{*\circ}_{2,ij}\right) & q^{*,-}_{1,ij} \leq q^{*,+}_{1,ij} \wedge q^{*,-}_{2,ij} > q^{*,+}_{2,ij} \\ \left(\overline{q}^{*\circ}_{1,ij}, \overline{q}^{*\circ}_{2,ij}\right) & q^{*,-}_{1,ij} > q^{*,+}_{1,ij} \wedge q^{*,-}_{2,ij} > q^{*,+}_{2,ij} \end{cases} \tag{26}$$

where $i, j = 1, 2, \dots, n$.

Example 4.1. Let $X = \{x_1, x_2, x_3, x_4\}$ be the object set. The incomplete IVIMPR $\tilde{R}$ on X is defined as:

$$\tilde{R} = \begin{pmatrix} ([1,1],[1,1]) & x & ([x,2],[\frac{1}{5},\frac{1}{3}]) & ([\frac{1}{4},\frac{1}{3}],[1,2]) \\ x & ([1,1],[1,1]) & ([\frac{1}{6},\frac{1}{4}],x) & (x,[\frac{1}{2},1]) \\ ([\frac{1}{5},\frac{1}{3}],[x,2]) & (x,[\frac{1}{6},\frac{1}{4}]) & ([1,1],[1,1]) & ([3,5],x) \\ ([1,2],[\frac{1}{4},\frac{1}{3}]) & ([\frac{1}{2},1],x) & (x,[3,5]) & ([1,1],[1,1]) \end{pmatrix}.$$

Using model (M-4), the missing values are determined as follows: $\tilde{r}_{12} = ([0.6667, 0.6667], [1.4815, 1.5])$, $\rho^{-}_{13} = 0.1111$, $\overline{\sigma}_{23} = [0.75, 0.75]$, $\overline{\rho}_{24} = [0.3953, 0.5]$, $\overline{\sigma}_{34} = [0.2, 0.2]$. For the complete IVIMPR $\tilde{R}$, we apply model (M-3) to judge its consistency, by which we have $\varphi^{*} = 20.6177$. From model (M-5), we obtain

$$\overline{Q}_1 = \begin{pmatrix} [1,1] & [0.6667,0.6667] & [0.1111,2] & [0.25,0.3333] \\ [1.5,1.5] & [1,1] & [0.1667,0.25] & [0.3953,0.5] \\ [9,2] & [6,4] & [1,1] & [3,5] \\ [4,3] & [2.5297,2] & [0.3333,0.2] & [1,1] \end{pmatrix},$$

$$\overline{Q}_2 = \begin{pmatrix} [1,1] & [0.675,0.6667] & [5,3] & [1,0.5] \\ [1.4815,1.5] & [1,1] & [1.3333,1.3333] & [2,1] \\ [0.2,0.3333] & [0.75,0.75] & [1,1] & [5,5] \\ [1,2] & [0.5,1] & [0.2,0.2] & [1,1] \end{pmatrix}.$$

From model (M-8), the obtained consistent QIVMPRs are

$$\overline{Q}^{*}_1 = \begin{pmatrix} [1,1] & [0.667,0.667] & [0.111,0.167] & [0.25,0.333] \\ [1.5,1.5] & [1,1] & [0.167,0.25] & [0.375,0.5] \\ [9,6] & [6,4] & [1,1] & [2.25,2] \\ [4,3] & [2.666,2] & [0.444,0.5] & [1,1] \end{pmatrix},$$

$$\overline{Q}^{*}_2 = \begin{pmatrix} [1,1] & [0.675,0.667] & [0.548,0.297] & [1.35,0.667] \\ [1.482,1.5] & [1,1] & [0.812,0.444] & [2,1] \\ [1.824,3.372] & [1.232,2.25] & [1,1] & [2.464,2.25] \\ [0.741,1.5] & [0.5,1] & [0.406,0.444] & [1,1] \end{pmatrix}.$$

According to $\overline{Q}^{*}_1$ and $\overline{Q}^{*}_2$, the consistent IVMPRs are

$$\overline{B}^{*}_1 = \begin{pmatrix} [1,1] & [0.667,0.667] & [0.111,0.167] & [0.25,0.333] \\ [1.5,1.5] & [1,1] & [0.167,0.25] & [0.375,0.5] \\ [6,9] & [4,6] & [1,1] & [2,2.25] \\ [3,4] & [2,2.666] & [0.444,0.5] & [1,1] \end{pmatrix},$$

$$\overline{B}_2^* = \begin{pmatrix} [1,1] & [0.667, 0.675] & [0.297, 0.548] & [0.667, 1.35] \\ [1.482, 1.5] & [1,1] & [0.444, 0.812] & [1,2] \\ [1.824, 3.372] & [1.232, 2.25] & [1,1] & [2.25, 2.464] \\ [0.741, 1.5] & [0.5,1] & [0.406, 0.444] & [1,1] \end{pmatrix}.$$

Furthermore, the consistent IVIMPR is following unnumbered equation which is given in Box I. It is worth noting that only one reference (Sahu et al., 2018) considers incomplete IVIMPR. However, it cannot be applied in this example because $B^{(1)} = \begin{pmatrix} 1 & \rho_{12}^- & \rho_{13}^- & \frac{1}{4} \\ \frac{1}{\rho_{12}^-} & 1 & \frac{1}{\rho_{12}^6} & \rho_{24}^- \\ \frac{1}{\rho_{13}^-} & 6 & 1 & 3 \\ 4 & \frac{1}{\rho_{24}^-} & \frac{1}{3} & 1 \end{pmatrix}$ following the incomplete IVIMPR $\tilde{R}$, and we cannot derive the determined value for ρ_{12}^- using equation (7) in Sahu et al. (2018).

5. Group decision making with IVIMPRs

In many situations, more than one DM is needed to make decisions for a practical problem, which is known as GDM (Liu et al., 2017; Pérez et al., 2010). In GDM, an important goal is to achieve solutions with a high consensus level. So, adequate consensus reaching processes based on consensus measures and sometimes also on consistency measures are developed.

This section considers GDM with IVIMPRs. To do this, the first part focuses on consensus, and the second part offers an algorithm to GDM with IVIMPRs.

5.1. Consensus analysis

To measure the agreement degree of individual opinions for the final ranking, consensus analysis is necessary. Suppose that there exist n objects $X = \{x_1, x_2, \ldots, x_n\}$, which are evaluated by m DMs $E = \{e_1, e_2, \ldots, e_m\}$. Let $\tilde{R}^h = (\tilde{r}_{ij}^h)_{n \times n}$ be the individual IVIMPR offered by the DM e_h, where $\tilde{r}_{ij}^h = (\overline{\rho}_{ij}^h, \overline{\sigma}_{ij}^h) = ([\rho_{ij}^{h,-}, \rho_{ij}^{h,+}], [\sigma_{ij}^{h,-}, \sigma_{ij}^{h,+}])$, $i, j = 1, 2, \ldots, n$; $h = 1, 2, \ldots, m$.

Definition 15. Let $\tilde{R}^h = (\tilde{r}_{ij}^h)_{n \times n}$ be an individual IVIMPR, and let $\overline{Q}_1^{h,*} = (\overline{q}_{1,ij}^{h,*})_{n \times n}$ and $\overline{Q}_2^{h,*} = (\overline{q}_{2,ij}^{h,*})_{n \times n}$ be its associated consistent QIVMPRs, where $h = 1, 2, \ldots, m$. Then, the comprehensive QIVMPRs $\overline{Q}_1^* = (\overline{q}_{1,ij}^*)_{n \times n}$ and $\overline{Q}_2^* = (\overline{q}_{2,ij}^*)_{n \times n}$ are defined as

$$\overline{q}_{1,ij}^* = \otimes_{h=1}^m \left(\overline{q}_{1,ij}^{h,*}\right)^{\omega_h} \quad \text{and} \quad \overline{q}_{2,ij}^* = \otimes_{h=1}^m \left(\overline{q}_{2,ij}^{h,*}\right)^{\omega_h} \tag{27}$$

where $i, j = 1, 2, \ldots, n$, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the weight vector on the DM set such that $\sum_{h=1}^m \omega_h = 1$ and $\omega_h \geq 0$, $h = 1, 2, \ldots, m$.

Property 2. Let $\overline{Q}_1^{h,*} = (\overline{q}_{1,ij}^{h,*})_{n \times n}$ and $\overline{Q}_2^{h,*} = (\overline{q}_{2,ij}^{h,*})_{n \times n}$ be the associated consistent QIVMPRs of $\tilde{R}^h = (\tilde{r}_{ij}^h)_{n \times n}$, $h = 1, 2, \ldots, m$, and let $\overline{Q}_1^* = (\overline{q}_{1,ij}^*)_{n \times n}$ and $\overline{Q}_2^* = (\overline{q}_{2,ij}^*)_{n \times n}$ be the associated comprehensive QIVMPRs as shown in Definition 15. Then, $\overline{Q}_1^*$ and $\overline{Q}_2^*$ are both consistent.

Proof. From Eq. (27) and the consistency of individual QIVMPRs, we have

$$\overline{q}_{1,ik}^* \otimes \overline{q}_{1,kj}^* = \left(\otimes_{h=1}^m \left(\overline{q}_{1,ik}^{h,*}\right)^{\omega_h}\right) \otimes \left(\otimes_{h=1}^m \left(\overline{q}_{1,kj}^{h,*}\right)^{\omega_h}\right)$$
$$= \otimes_{h=1}^m \left(\overline{q}_{1,ik}^{h,*} \otimes \overline{q}_{1,kj}^{h,*}\right)^{\omega_h} = \otimes_{h=1}^m \left(\overline{q}_{1,ij}^{h,*}\right)^{\omega_h} = \overline{q}_{1,ij}^*$$

Thus, $\overline{Q}_1^*$ is consistent. Similarly, we obtain the consistency of $\overline{Q}_2^*$.

Now, we apply consistent QIVMPRs to offer a consensus index.

Definition 16. Let $\tilde{R}^h = (\tilde{r}_{ij}^h)_{n \times n}$ be an individual IVIMPR and let $\overline{Q}_1^{h,*} = (\overline{q}_{1,ij}^{h,*})_{n \times n}$ and $\overline{Q}_2^{h,*} = (\overline{q}_{2,ij}^{h,*})_{n \times n}$ be its associated consistent QIVMPRs,

where $h = 1, 2, \ldots, m$. Then, the consensus index of $\tilde{R}^h = (\tilde{r}_{ij}^h)_{n \times n}$ is defined as:

$$COI(\tilde{R}^h) = 1 - \frac{1}{4n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left|\log_9(q_{1,ij}^{h,*,-}) - \log_9(q_{1,ij}^{*,-})\right|\right.$$
$$+ \left|\log_9(q_{1,ij}^{h,*,+}) - \log_9(q_{1,ij}^{*,+})\right|$$
$$+ \left.\left|\log_9(q_{2,ji}^{h,*,-}) - \log_9(q_{2,ji}^{*,-})\right| + \left|\log_9(q_{2,ji}^{h,*,+}) - \log_9(q_{2,ji}^{*,+})\right|\right), \tag{28}$$

where $\overline{Q}_1^* = (\overline{q}_{1,ij}^*)_{n \times n}$ and $\overline{Q}_2^* = (\overline{q}_{2,ij}^*)_{n \times n}$ are the comprehensively consistent QIVMPRs shown in Definition 15.

Note that $0 \leq COI(\tilde{R}^h) \leq 1$ for any individual IVIMPR $\tilde{R}^h$, $h = 1, 2, \ldots, m$, and

$$\begin{cases} 0 \leq \left|\log_9(q_{1,ij}^{h,*,-}) - \log_9(q_{1,ij}^{*,-})\right| \leq 2 \\ 0 \leq \left|\log_9(q_{1,ij}^{h,*,+}) - \log_9(q_{1,ij}^{*,+})\right| \leq 2 \\ 0 \leq \left|\log_9(q_{2,ji}^{h,*,-}) - \log_9(q_{2,ji}^{*,-})\right| \leq 2 \\ 0 \leq \left|\log_9(q_{2,ji}^{h,*,+}) - \log_9(q_{2,ji}^{*,+})\right| \leq 2. \end{cases}$$

Property 3. Let $\overline{Q}_1^{h,*} = (\overline{q}_{1,ij}^{h,*})_{n \times n}$ and $\overline{Q}_2^{h,*} = (\overline{q}_{2,ij}^{h,*})_{n \times n}$ be the consistent QIVMPRs obtained from the individual IVIMPR $\tilde{R}^h = (\tilde{r}_{ij}^h)_{n \times n}$, $h = 1, 2, \ldots, m$, and let $\overline{Q}_1^* = (\overline{q}_{1,ij}^*)_{n \times n}$ and $\overline{Q}_2^* = (\overline{q}_{2,ij}^*)_{n \times n}$ be the associated comprehensive QIVMPRs shown in Definition 16, where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the weight vector on the DM set such that $\sum_{h=1}^m \omega_h = 1$ and $\omega_h \geq 0$, $h = 1, 2, \ldots, m$. Let $\overline{Q}_1'^{h,*} = (\overline{q}_{1,ij}'^{h,*})_{n \times n}$ and $\overline{Q}_2'^{h,*} = (\overline{q}_{2,ij}'^{h,*})_{n \times n}$ be the adjusted QIVMPRs for the consistent QIVMPRs $\overline{Q}_1^{h,*}$ and $\overline{Q}_2^{h,*}$, where

$$\overline{q}_{1,ij}'^{h,*} = \left(\overline{q}_{1,ij}^{h,*}\right)^v \otimes \left(\overline{q}_{1,ij}^*\right)^{1-v} \quad \text{and} \quad \overline{q}_{2,ij}'^{h,*} = \left(\overline{q}_{2,ij}^{h,*}\right)^v \otimes \left(\overline{q}_{2,ij}^*\right)^{1-v} \tag{29}$$

$i, j = 1, 2, \ldots, n$, and $v \in (0, 1)$.
(i) The individual QIVMPRs $\overline{Q}_1'^{h,*} = (\overline{q}_{1,ij}'^{h,*})_{n \times n}$ and $\overline{Q}_2'^{h,*} = (\overline{q}_{2,ij}'^{h,*})_{n \times n}$ are consistent;
(ii) Let $\tilde{R}'^h = (\tilde{r}_{ij}'^h)_{n \times n}$ be the individual IVIMPR obtained from the QIVMPRs $\overline{Q}_1'^{h,*} = (\overline{q}_{1,ij}'^{h,*})_{n \times n}$ and $\overline{Q}_2'^{h,*} = (\overline{q}_{2,ij}'^{h,*})_{n \times n}$, we have $COI(\tilde{R}'^h) \geq COI(\tilde{R}^h)$.

Proof. For (i): Property 2 shows that the comprehensive QIVMPRs $\overline{Q}_1^* = (\overline{q}_{1,ij}^*)_{n \times n}$ and $\overline{Q}_2^* = (\overline{q}_{2,ij}^*)_{n \times n}$ are consistent. Similar to Property 2, we derive (i).

For (ii): For each pair of (i, j) with $i < j$, we have

$$\left|\log_9(q_{1,ij}'^{h,*,-}) - \log_9(\Pi_{l=1,l\neq h}^m q_{1,ij}^{*,l,-} \times q_{1,ij}'^{h,*,-})\right|$$
$$= \left|\log_9\left(\left(q_{1,ij}^{h,*,-}\right)^v \times \left(q_{1,ij}^{*,-}\right)^{1-v}\right) - \log_9\left(\left(\Pi_{l=1,l\neq h}^m q_{1,ij}^{*,l,-}\right)^{\omega_l}\right.\right.$$
$$\left.\left.\times \left(\left(q_{1,ij}^{h,*,-}\right)^v \otimes \left(q_{1,ij}^{*,-}\right)^{1-v}\right)^{\omega_h}\right)\right|$$
$$= \left|v\log_9\left(q_{1,ij}^{h,*,-}\right) + (1-v)\log_9\left(q_{1,ij}^{*,-}\right) - \log_9\left(q_{1,ij}^{*,-}\right)\right.$$
$$\left.-(v-1)\omega_h\log_9\left(q_{1,ij}^{h,*,-}\right) - (1-v)\omega_h\log_9\left(q_{1,ij}^{*,-}\right)\right|$$
$$\leq \left|v\log_9\left(q_{1,ij}^{h,*,-}\right) - v\log_9\left(q_{1,ij}^{*,-}\right)\right|$$
$$+ \left|(1-v)\omega_h\log_9\left(q_{1,ij}^{h,*,-}\right) - (1-v)\omega_h\log_9\left(q_{1,ij}^{*,-}\right)\right|$$
$$\leq v\left|\log_9\left(q_{1,ij}^{h,*,-}\right) - \log_9\left(q_{1,ij}^{*,-}\right)\right| + (1-v)\left|\log_9\left(q_{1,ij}^{h,*,-}\right) - \log_9\left(q_{1,ij}^{*,-}\right)\right|$$
$$= \left|\log_9\left(q_{1,ij}^{h,*,-}\right) - \log_9\left(q_{1,ij}^{*,-}\right)\right|.$$

$$\tilde{R}^* = \begin{pmatrix} ([1,1],[1,1]) & ([0.667,0.667],[1.482,1.5]) & ([0.111,0.167],[1.824,3.372]) & ([0.25,0.333],[0.741,1.5]) \\ ([1.482,1.5],[0.667,0.667]) & ([1,1],[1,1]) & ([0.167,0.25],[1.232,2.25]) & ([0.375,0.5],[0.5,1]) \\ ([1.824,3.372],[0.1111,0.167]) & ([1.232,2.25],[0.167,0.25]) & ([1,1],[1,1]) & ([2,2.25],[0.406,0.444]) \\ ([0.741,1.5],[0.25,0.333]) & ([0.5,1],[0.375,0.5]) & ([0.406,0.444],[2,2.25]) & ([1,1],[1,1]) \end{pmatrix}$$

Box I.

Similarly, we derive

$$\begin{cases} \left| \log_9(q_{1,ij}'^{h,*,+}) - \log_9(\Pi_{l=1,l\neq h}^m q_{1,ij}^{*,l,+} \times q_{1,ij}'^{h,*,+}) \right| \\ \quad \leq \left| \log_9\left(q_{1,ij}^{h,*,+}\right) - \log_9\left(q_{1,ij}^{*,+}\right) \right| \\ \left| \log_9(q_{2,ji}'^{h,*,-}) - \log_9(\Pi_{l=1,l\neq h}^m q_{2,ji}^{*,l,-} \times q_{2,ji}'^{h,*,-}) \right| \\ \quad \leq \left| \log_9\left(q_{2,ji}^{h,*,-}\right) - \log_9\left(q_{2,ji}^{*,-}\right) \right| \\ \left| \log_9(q_{2,ji}'^{h,*,+}) - \log_9(\Pi_{l=1,l\neq h}^m q_{2,ji}^{*,l,+} \times q_{2,ji}'^{h,*,+}) \right| \\ \quad \leq \left| \log_9\left(q_{2,ji}^{h,*,+}\right) - \log_9\left(q_{2,ji}^{*,+}\right) \right|. \end{cases}$$

Eq. (28) shows that $COI(\tilde{R}'^h) \geq COI(\tilde{R}^h)$.

Properties 2 and 3 show that the comprehensive QIVMPRs $\overline{Q}'_1 = (\overline{q}'^*_{1,ij})_{n\times n}$ and $\overline{Q}'_2 = (\overline{q}'^*_{2,ij})_{n\times n}$ are consistent from the individual consistent QIVMPRs.

Definition 16 shows that the weights of the DMs are needed for calculating comprehensively consistent QIVMPRs. However, the weighting information may be unknown. To address this problem, we introduce the following method.

Definition 17. Let $\tilde{R}^1 = (\tilde{r}^1_{ij})_{n\times n}$ and $\tilde{R}^2 = (\tilde{r}^2_{ij})_{n\times n}$ be any two IVIMPRs, and let $\begin{cases} \overline{B}^{1,*}_1 = (\overline{b}^{1,*}_{1,ij})_{n\times n} \\ \overline{B}^{1,*}_2 = (\overline{b}^{1,*}_{2,ij})_{n\times n} \end{cases}$ and $\begin{cases} \overline{B}^{2,*}_1 = (\overline{b}^{2,*}_{1,ij})_{n\times n} \\ \overline{B}^{2,*}_2 = (\overline{b}^{2,*}_{2,ij})_{n\times n} \end{cases}$ be their associated consistent QIVMPRs. Then, the distance between the IVIMPRs $\tilde{R}^1$ and $\tilde{R}^2$ is defined as:

$$D\left(\tilde{R}^1, \tilde{R}^2\right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left| b^{1,*,-}_{1,ij} - b^{2,*,-}_{1,ij} \right| + \left| b^{1,*,+}_{1,ij} - b^{2,*,+}_{1,ij} \right| + \left| b^{1,*,-}_{2,ji} - b^{2,*,-}_{2,ji} \right| + \left| b^{1,*,+}_{2,ji} - b^{2,*,+}_{2,ji} \right| \right). \tag{30}$$

Property 4. *Let $\tilde{R}^1 = (\tilde{r}^1_{ij})_{n\times n}$, $\tilde{R}^2 = (\tilde{r}^2_{ij})_{n\times n}$ and $\tilde{R}^3 = (\tilde{r}^3_{ij})_{n\times n}$ be any three IVIMPRs. Then, their distance measure defined by Eq. (30) has the following characteristics:*
(i) $D\left(\tilde{R}^1, \tilde{R}^2\right) = 0 \Leftrightarrow \tilde{R}^1 = \tilde{R}^2$;
(ii) $D\left(\tilde{R}^1, \tilde{R}^2\right) = D\left(\tilde{R}^2, \tilde{R}^1\right)$;
(iii) $D\left(\tilde{R}^1, \tilde{R}^2\right) + D\left(\tilde{R}^2, \tilde{R}^3\right) \geq D\left(\tilde{R}^1, \tilde{R}^3\right)$.

Proof. From Eq. (30), it is easy to derive the conclusion.

Next, we build a distance measure-based model to determine the weights of the DMs.

$$\min \sum_{h=1}^{m} \left(\left(\sum_{l=1,l\neq h}^{m} D\left(\tilde{R}^h, \tilde{R}^l\right) \right) \omega_h \right)$$

$$s.t. \begin{cases} \sum_{h=1}^{m} \omega_h = 1 \\ \omega_h \in [\omega_h^-, \omega_h^+], h = 1, 2, \ldots, m \\ \omega_h \geq 0, h = 1, 2, \ldots, m, \end{cases} \tag{M-9}$$

where $[\omega_h^-, \omega_h^+]$ is the known weight information of the DM e_h, and $D\left(\tilde{R}^h, \tilde{R}^l\right)$ is distance measure between the individual IVIMPRs $\tilde{R}^h$ and $\tilde{R}^l$ shown in Eq. (30).

5.2. An algorithm to GDM with IVIMPRs

From the above discussion, this subsection introduces an algorithm to GDM with IVIMPRs based on the consistency and consensus analysis.

Step 1: Let $\tilde{R}^h = (\tilde{r}^h_{ij})_{n\times n}$ be the individual IVIMPR offered by e_h, $h = 1, 2, \ldots, m$. If incomplete IVIMPRs exist, model (M-4) is used to determine missing values. Otherwise, go to the next step;

Step 2: For each complete IVIMPR $\tilde{R}^h = (\tilde{r}^h_{ij})_{n\times n}$, model (M-3) is applied to judge the consistency. Then, model (M-5) is adopted to derive the associated QIVMPRs $\overline{Q}^h_1 = (\overline{q}^h_{1,ij})_{n\times n}$ and $\overline{Q}^h_2 = (\overline{q}^h_{2,ij})_{n\times n}$, $h = 1, 2, \ldots, m$;

Step 3: When the individual QIVMPRs $\overline{Q}^h_1$ and $\overline{Q}^h_2$ are consistent, go to Step 4. Otherwise, model (M-8) is adopted to derive individual consistent QIVMPRs $\overline{Q}^{h,*}_1$ and $\overline{Q}^{h,*}_2$, $h = 1, 2, \ldots, m$;

Step 4: Model (M-9) is adopted to determine the weights of the DMs;

Step 5: According to $\overline{Q}^{h,*}_1$ and $\overline{Q}^{h,*}_2$, $h = 1, 2, \ldots, m$, the comprehensively consistent QIVMPRs $\overline{Q}^*_1 = (\overline{q}^*_{1,ij})_{n\times n}$ and $\overline{Q}^*_2 = (\overline{q}^*_{2,ij})_{n\times n}$ can be obtained;

Step 6: Let π^* be the threshold of consensus. If $COI(\tilde{R}^h) \geq \pi^*$, $h = 1, 2, \ldots, m$, then turn to Step 8. Otherwise, go to next step;

Step 7: Let $COI(\tilde{R}^h) = \min_{1\leq l\leq m} COI(\tilde{R}^l) < \pi^*$. Property 3 is used to adjust the consensus of $\overline{Q}^{h,*}_1$ and $\overline{Q}^{h,*}_2$, and return to Step 5;

Step 8: Based on the comprehensively consistent QIVMPRs $\overline{Q}^*_1 = (\overline{q}^*_{1,ij})_{n\times n}$ and $\overline{Q}^*_2 = (\overline{q}^*_{2,ij})_{n\times n}$, the comprehensively consistent IVIMPR $\tilde{R}^* = (\tilde{r}^*_{ij})_{n\times n}$ is obtained;

Step 9: The following equation is applied to calculate the interval-valued intuitionistic multiplicative priority weights:

$$\tilde{w}_i = \left(\left[\sqrt[n]{\Pi_{j=1}^n \rho_{ij}^{*,-}}, \sqrt[n]{\Pi_{j=1}^n \rho_{ij}^{*,+}} \right], \left[\sqrt[n]{\Pi_{j=1}^n \sigma_{ij}^{*,-}}, \sqrt[n]{\Pi_{j=1}^n \sigma_{ij}^{*,+}} \right] \right)$$

$$i = 1, 2, \ldots, n. \tag{31}$$

Step 10: Eq. (4) is used to rank $\tilde{w}_i$, $i = 1, 2, \ldots, n$, by which the ranking of objects $x_1, x_2, \ldots, x_n$ is derived.

6. A case study

Nowadays, the competition among enterprises has transformed into the competition among the enterprises' supply chains. The suppliers are the "lionhead" of the whole supply chain, which influences the success of downstream manufacturers in delivery, product quality, lead time, inventory level, and product design etc. The quality and price of products supplied by suppliers determine the quality and price of the final consumption goods and affect the market competitiveness, market share and market viability of the final products as well as the core competitiveness of each component of the supply chain. Therefore, how to choose the suitable supplier is an important researching topic. A Chinese car company plans to select a steel supplier. Through investigation and study, four steel producers were selected as potential suppliers: Hebei iron & steel group corporation (HBIS), Shougang group corporation (SHOUGANG), Baosteel group corporation (BAOSTEEL), and Anshan iron and steel group corporation (ANSTEEL). To determine the final steel supplier, four experts are invited to evaluate these four steel corporations. Because there are many factors to influence the judgments of the experts, they are permitted to apply intervals on

Saaty's [1/9, 9] scale to denote their uncertainties. Furthermore, the experts can offer the uncertain preferred and non-preferred judgments simultaneously. When they are unable or unwilling to make judgments for some comparisons, missing information is allowed too. On basis of their expertise and personal preferences, the individual IVIMPRs are offered as shown in Tables 1–4. Taking the comparison between HBIS and SHOUGANG for example, the first expert thinks that the preferred degree of HBIS over SHOUGANG is between 3 to 5 times, while he/she considers that the preferred degree of SHOUGANG over HBIS is between 1/8 to 1/6 times rather than between 1/5 to 1/3 times due to the inconsistency of subjective judgments. To express the above information, IVIMVs are good choices and the above judgments can be denoted as ([3, 5], [1/8, 1/6]). Furthermore, the first expert is unable or unwilling to give his/her comparison judgment between BAOSTEEL and ANSTEEL. To obtain the same preference structure as other judgments, we here employ IVIMVs, denoted by ($[\rho_{34}^{1,-}, \rho_{34}^{1,+}], [\sigma_{34}^{1,-}, \sigma_{34}^{1,+}]$).

Suppose that the known weighting information of the experts is $\omega_h \in [0.2, 0.3], h = 1, 2, 3, 4$. To rank these four steel corporations and select the best one(s), the following procedure is needed:

Step 1: Because all individual IVIMPRs are incomplete, we apply model (M-4) to derive complete IVIMPRs. Please see Tables 5–8.

Step 2: For each complete IVIMPR, the objective function values of model (M-3) are

$$\varphi^*(\tilde{R}^1) = 14.9787, \varphi^*(\tilde{R}^2) = 15.8535, \varphi^*(\tilde{R}^3) = 8.0923, \varphi^*(\tilde{R}^4) = 9.8544,$$

namely, none of them is consistent. Using model (M-5), individual QIVMPRs can be obtained. Taking the individual IVIMPR $\tilde{R}^1$ for example, the individual QIVMPRs are

$$\overline{Q}_1^1 = \begin{pmatrix} [1,1] & [3,5] & [0.5,1] & [6,7] \\ [0.333,0.2] & [1,1] & [0.225,0.236] & [2,3] \\ [2,1] & [4.444,4.237] & [1,1] & [9,7.884] \\ [0.167,0.143] & [0.5,0.333] & [0.111,0.127] & [1,1] \end{pmatrix}$$

$$\overline{Q}_2^1 = \begin{pmatrix} [1,1] & [8,6] & [4,2] & [9,7] \\ [0.125,0.167] & [1,1] & [5,3] & [3,3] \\ [0.25,0.5] & [0.2,0.333] & [1,1] & [9,9] \\ [0.111,0.143] & [0.333,0.333] & [0.111,0.111] & [1,1] \end{pmatrix}.$$

Step 3: We adopt model (M-8) to derive individual consistent QIVMPRs. Taking the individual QIVMPRs $\overline{Q}_1^1$ and $\tilde{Q}_2^1$ for instance, the individual consistent QIVMPRs are

$$\overline{Q}_1^{1*} = \begin{pmatrix} [1,1] & [3,4.237] & [0.675,1] & [6,7.884] \\ [0.333,0.236] & [1,1] & [0.225,0.236] & [2,1.861] \\ [1.482,1] & [4.444,4.237] & [1,1] & [8.889,7.884] \\ [0.167,0.127] & [0.5,0.537] & [0.112,0.127] & [1,1] \end{pmatrix}$$

$$\overline{Q}_2^{1*} = \begin{pmatrix} [1,1] & [8,5.92] & [2.703,2] & [24.023,17.777] \\ [0.125,0.169] & [1,1] & [0.338,0.338] & [3.003,3.003] \\ [0.37,0.5] & [2.96,2.96] & [1,1] & [8.889,8.889] \\ [0.042,0.056] & [0.333,0.333] & [0.112,0.112] & [1,1] \end{pmatrix}.$$

Step 4: Using model (M-9), the weights of the experts are $\omega_1 = 0.2$, $\omega_2 = 0.3$, $\omega_3 = 0.2$, and $\omega_4 = 0.3$.

Step 5: Based on individual consistent QIVMPRs and the weights of the experts, the comprehensively consistent QIVMPRs are

$$\overline{Q}_1^* = \begin{pmatrix} [1,1] & [2.049,2.373] & [1.303,1.705] & [4.243,6.069] \\ [0.488,0.421] & [1,1] & [0.636,0.718] & [2.071,2.557] \\ [0.767,0.587] & [1.572,1.392] & [1,1] & [3.256,3.56] \\ [0.236,0.165] & [0.483,0.391] & [0.307,0.281] & [1,1] \end{pmatrix}$$

$$\overline{Q}_2^* = \begin{pmatrix} [1,1] & [4.88,4.017] & [4.68,3.101] & [18.911,12.706] \\ [0.205,0.249] & [1,1] & [0.959,0.772] & [3.875,3.163] \\ [0.214,0.323] & [1.043,1.296] & [1,1] & [4.041,4.098] \\ [0.053,0.078] & [0.258,0.316] & [0.247,0.244] & [1,1] \end{pmatrix}.$$

Step 6: Let $\pi^* = 0.9$. By Eq. (28), we have

$$COI(\tilde{R}^1) = 0.869, COI(\tilde{R}^2) = 0.917, COI(\tilde{R}^3) = 0.806, COI(\tilde{R}^4) = 0.901.$$

Because $COI(\tilde{R}^3) = \min_{1\leq h\leq 4} COI(\tilde{R}^h) = 0.806 < 0.9$, we need to adjust individual consistent QIVMPRs. After adjusting four times for individual consistent QIVMPRs $\overline{Q}_1^{3*}$ and $\overline{Q}_2^{3*}$ and one time for individual consistent QIVMPRs $\overline{Q}_1^{1*}$ and $\overline{Q}_2^{1*}$, we have

$$\overline{Q}_1^{\prime 1*} = \begin{pmatrix} [1,1] & [2.814,3.817] & [0.754,1.078] & [5.647,7.484] \\ [0.355,0.262] & [1,1] & [0.268,0.282] & [2.007,1.961] \\ [1.327,0.928] & [3.732,3.541] & [1,1] & [7.49,6.943] \\ [0.177,0.134] & [0.498,0.51] & [0.133,0.144] & [1,1] \end{pmatrix},$$

$$\overline{Q}_2^{\prime 1*} = \begin{pmatrix} [1,1] & [7.453,5.615] & [2.976,2.147] & [23.401,16.873] \\ [0.134,0.178] & [1,1] & [0.399,0.383] & [3.14,3.005] \\ [0.336,0.466] & [2.504,2.615] & [1,1] & [7.864,7.858] \\ [0.043,0.059] & [0.318,0.333] & [0.127,0.127] & [1,1] \end{pmatrix};$$

$$\overline{Q}_1^{\prime 3*} = \begin{pmatrix} [1,1] & [1.71,2.001] & [1.777,2.724] & [3.729,6.044] \\ [0.584,0.5] & [1,1] & [1.04,1.362] & [2.181,3.021] \\ [0.562,0.367] & [0.962,0.734] & [1,1] & [2.098,2.219] \\ [0.268,0.167] & [0.459,0.33] & [0.476,0.451] & [1,1] \end{pmatrix},$$

$$\overline{Q}_2^{\prime 3*} = \begin{pmatrix} [1,1] & [3.222,2.784] & [5.714,3.964] & [13.728,10.173] \\ [0.311,0.359] & [1,1] & [1.773,1.424] & [4.26,3.655] \\ [0.175,0.253] & [0.563,0.702] & [1,1] & [2.403,2.565] \\ [0.073,0.099] & [0.234,0.273] & [0.416,0.389] & [1,1] \end{pmatrix}.$$

Furthermore, associated comprehensively consistent QIVMPRs are

$$\overline{Q}_1^{\prime *} = \begin{pmatrix} [1,1] & [2.15,2.462] & [1.2,1.478] & [4.378,6.015] \\ [0.465,0.406] & [1,1] & [0.558,0.6] & [2.036,2.443] \\ [0.833,0.677] & [1.792,1.666] & [1,1] & [3.649,4.07] \\ [0.228,0.167] & [0.491,0.409] & [0.274,0.246] & [1,1] \end{pmatrix},$$

$$\overline{Q}_2^{\prime *} = \begin{pmatrix} [1,1] & [5.535,4.498] & [4.46,2.895] & [20.958,13.553] \\ [0.181,0.222] & [1,1] & [0.806,0.644] & [3.786,3.014] \\ [0.224,0.346] & [1.241,1.554] & [1,1] & [4.699,4.682] \\ [0.048,0.074] & [0.264,0.332] & [0.213,0.214] & [1,1] \end{pmatrix}.$$

From the above adjusted individual consistent QIVMPRs, we derive

$$COI(\tilde{R}^1) = 0.908, COI(\tilde{R}^2) = 0.924, COI(\tilde{R}^3) = 0.910, COI(\tilde{R}^4) = 0.904.$$

Step 7: From comprehensively consistent QIVMPRs $\overline{Q}_1^{\prime *}$ and $\overline{Q}_2^{\prime *}$, the comprehensively consistent IVIMPR is obtained as shown in Table 9.

Step 8: From $\tilde{R}^*$, the interval-valued intuitionistic multiplicative priority weights are

$$\tilde{w}_1 = ([1.833, 2.163], [0.21, 0.275]), \ \tilde{w}_2 = ([0.673, 0.755], [0.916, 1.062]),$$
$$\tilde{w}_3 = ([1.004, 1.216], [0.615, 0.66]), \ \tilde{w}_4 = ([0.228, 0.269], [2.388, 2.781]).$$

Step 9: From Eq. (4), we have $S(\tilde{w}_1) = 8.292, S(\tilde{w}_2) = 0.723, S(\tilde{w}_3) = 1.735, S(\tilde{w}_4) = 0.095$.

Thus, the ranking is HBIS > SHOUGANG > BAOSTEEL > ANSTEEL, and HBIS is the best choice.

It is noticeable that the above ranking results are derived from completely consistent IVIMPR which ensures the logicality. One the other hand, this ranking represents no less than 90% consensual degrees of the experts. From the final scores of these four steel companies, HBIS is the best choice, while ANSTEEL is the worst. With the increasing competition among steel companies, each enterprise must enhance the research and development of new products and improve product quality. Meanwhile, they should reduce production costs by optimizing their industrial institutions. Then, they are not eliminated by the market in the fierce competition.

Note that previous methods (Jiang et al., 2014; Liu et al., 2019; Zhang, 2017; Zhang and Prdrycz, 2019; Zhang et al., 2019) for decision making with interval-valued intuitionistic multiplicative information cannot be applied in this example, which do not consider incomplete

Table 1
Individual IVIMPR $\tilde{R}^1$ offered by the first expert.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([3, 5], [1/8,1/6])	([1/2, 1], [1/4,1/2])	([6, 7], [1/9,1/7])
SHOUGANG	([1/8,1/6], [3, 5])	([1, 1], [1,1])	([$\rho_{23}^{1,-},\rho_{23}^{1,+}$], [1/5, 1/3])	([2, 3], [$\sigma_{24}^{1,-},\sigma_{24}^{1,+}$])
BAOSTEEL	([1/4,1/2], [1/2, 1])	([1/5, 1/3], [$\sigma_{32}^{1,-},\sigma_{32}^{1,+}$])	([1, 1], [1,1])	([$\rho_{34}^{1,-},\rho_{34}^{1,+}$], [$\sigma_{34}^{1,-},\sigma_{34}^{1,+}$])
ANSTEEL	([1/9,1/7], [6, 7])	([$\rho_{42}^{1,-},\rho_{42}^{1,+}$], [2, 3])	([$\rho_{43}^{1,-},\rho_{43}^{1,+}$], [$\sigma_{43}^{1,-},\sigma_{43}^{1,+}$])	([1, 1], [1,1])

Table 2
Individual IVIMPR $\tilde{R}^2$ offered by the second expert.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([1, 2], [1/4,1/3])	([$\rho_{13}^{2,-},\rho_{13}^{2,+}$], [1/5, 1/3])	([2, 3], [1/5,1/4])
SHOUGANG	([1/4,1/3], [1, 2])	([1, 1], [1,1])	([1/2, 1], [1/4,1/2])	([$\rho_{24}^{2,-},\rho_{24}^{2,+}$], [1/3, 1/2])
BAOSTEEL	([1/5, 1/3], [$\sigma_{31}^{2,-},\sigma_{31}^{2,+}$])	([1/4,1/2], [1/2, 1])	([1, 1], [1,1])	([6, 8], [1/9,1/8])
ANSTEEL	([1/5,1/4], [2, 3])	([1/3, 1/2], [$\sigma_{42}^{2,-},\sigma_{42}^{2,+}$])	([1/9,1/8], [6, 8])	([1, 1], [1,1])

Table 3
Individual IVIMPR $\tilde{R}^3$ offered by the third expert.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([$\rho_{12}^{3,-},\rho_{12}^{3,+}$], [$\sigma_{12}^{3,-},\sigma_{12}^{3,+}$])	([4, 6], [1/8,1/6])	([3, 6], [1/8,1/7])
SHOUGANG	([$\rho_{21}^{3,-},\rho_{21}^{3,+}$], [$\sigma_{21}^{3,-},\sigma_{21}^{3,+}$])	([1, 1], [1,1])	([2, 4], [1/5,1/4])	([$\rho_{24}^{3,-},\rho_{24}^{3,+}$], [$\sigma_{24}^{3,-},\sigma_{24}^{3,+}$])
BAOSTEEL	([1/8,1/6], [4, 6])	([1/5,1/4], [2, 4])	([1, 1], [1,1])	([1, 3], [1/5,1/3])
ANSTEEL	([1/8,1/7], [3, 6])	([$\rho_{42}^{3,-},\rho_{42}^{3,+}$], [$\sigma_{42}^{3,-},\sigma_{42}^{3,+}$])	([1/5,1/3], [1, 3])	([1, 1], [1,1])

Table 4
Individual IVIMPR $\tilde{R}^4$ offered by the fourth expert.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([7, 8], [1/9,1/8])	([5, 6], [1/8,1/6])	([$\rho_{14}^{4,-},\rho_{14}^{4,+}$], [$\sigma_{14}^{4,-},\sigma_{14}^{4,+}$])
SHOUGANG	([1/9,1/8], [7, 8])	([1, 1], [1,1])	([$\rho_{23}^{4,-},\rho_{23}^{4,+}$], [$\sigma_{23}^{4,-},\sigma_{23}^{4,+}$])	([2, 3], [1/5,1/4])
BAOSTEEL	([1/8,1/6], [5, 6])	([$\rho_{32}^{4,-},\rho_{32}^{4,+}$], [$\sigma_{32}^{4,-},\sigma_{32}^{4,+}$])	([1, 1], [1,1])	([3, 5], [1/7,1/6])
ANSTEEL	([$\rho_{41}^{4,-},\rho_{41}^{4,+}$], [$\sigma_{41}^{4,-},\sigma_{41}^{4,+}$])	([1/5,1/4], [2, 3])	([1/7,1/6], [3, 5])	([1, 1], [1,1])

Table 5
Completely individual IVIMPR $\tilde{R}^1$.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([3, 5], [0.125,0.167])	([0.5, 1], [0.25,00.5])	([6, 7], [0.111,0.143])
SHOUGANG	([0.125,0.167], [3, 5])	([1, 1], [1,1])	([0.225, 0.236], [0.2, 0.333])	([2, 3], [0.333,0.333])
BAOSTEEL	([0.25,0.5], [0.5, 1])	([0.2, 0.333], [0.225, 0.236])	([1, 1], [1,1])	([7.884, 9], [0.111, 0.111])
ANSTEEL	([0.111,0.143], [6, 7])	([0.333,0.333], [2, 3])	([0.111, 0.111], [7.884, 9])	([1, 1], [1,1])

Table 6
Completely individual IVIMPR $\tilde{R}^2$.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([1, 2], [0.25,0.333])	([0.408, 0.866], [0.2,0.333])	([2, 3], [0.2,0.25])
SHOUGANG	([0.25,0.333], [1, 2])	([1, 1], [1,1])	([0.5, 1], [0.25,0.5])	([2, 2], [0.333,0.5])
BAOSTEEL	([0.2,0.333], [0.408, 0.866])	([0.25,0.5], [0.5, 1])	([1, 1], [1,1])	([6, 8], [0.111,0.125])
ANSTEEL	([0.2,0.25], [2, 3])	([0.333,0.5], [2, 2])	([0.111,0.125], [6, 8])	([1, 1], [1,1])

case. Sahu et al. (2018) only offered two methods to ascertain missing judgment and disregarded how to calculate the priority vector and rank objects based on completely individual IVIMPRs. Furthermore, the authors did not research how to obtain (acceptably) consistent IVIMPRs from inconsistent ones. Therefore, we cannot use Sahu et al.'s method in this case study either. This case study shows that the new method extends the application of IVIMPRs.

7. Conclusion

To address the vagueness of decision-making problems, various types of fuzzy sets are proposed that are based on different point of views. Considering the situation where the DMs may want to denote their asymmetrical uncertain preferred and non-preference judgements simultaneously, this paper introduces a consistency-and-consensus-analysis-based method for GDM with IVIMPRs that can cope with incomplete and inconsistent cases. To do this, we mainly do the following jobs: defining a consistency concept, establishing a model for judging the consistency, building a model for determining missing

values, offering a method to derive consistent IVIMPRs, analyzing the consensus, and constructing a model to obtain the weights of the DMs. To show the specific application of the new results, a practical decision-making problem about selecting the most suitable steel supplier is provided. Comparing with previous methods (Jiang et al., 2014; Zhang, 2017), the main advantages of our method include: (i) the new consistency concept is more reasonable than previous ones that avoids their limitations; (ii) it can address decision making with missing information, where ignored objects exist; (iii) it is based on the consistency and consensus analysis that guarantees the rationality and representativeness of ranking. However, new method seems to be more complex than previous ones (Jiang and Xu, 2014; Zhang, 2017), which needs a series of decision-making steps. However, with the help of computer, this issue is easy to solve.

Note that one-to-one mapping exists between IVIMPRs and IVIFPRs (Xu, 2007a). This conclusion can be easily derived as IMFPRs (Saaty and Vargas, 1987) and interval fuzzy preference relations (Xu and Yager, 2009). This paper focuses on the theory and application of IVIMVs in setting of preference relations, and we shall continue to

Table 7
Completely individual IVIMPR $\tilde{R}^3$.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([1.26, 1.5], [0.625,0.633])	([4, 6], [0.125,0.167])	([3, 6], [0.125,0.167])
SHOUGANG	([0.625,0.633],[1.26, 1.5])	([1, 1], [1,1])	([2, 4], [0.2,0.25])	([3.78, 4], [0.2,0.238])
BAOSTEEL	([0.125,0.167], [4, 6])	([0.2,0.25], [2, 4])	([1, 1], [1,1])	([1, 3], [0.2,0.333])
ANSTEEL	([0.125,0.167], [3, 6])	([0.2,0.238], [3.78, 4])	([0.2,0.333], [1, 3])	([1, 1], [1,1])

Table 8
Completely individual IVIMPR $\tilde{R}^4$.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([7, 8], [0.111,0.125])	([5, 6], [0.125, 0.167])	([9, 9], [0.111, 0.111])
SHOUGANG	([0.111,0.125], [7, 8])	([1, 1], [1,1])	([0.671, 0.69], [1.225, 1.449])	([2, 3], [0.2,0.25])
BAOSTEEL	([0.125, 0.167], [5, 6])	([1.225, 1.449], [0.671, 0.69])	([1, 1], [1,1])	([3, 5], [0.143,0.167])
ANSTEEL	([0.111, 0.111], [9, 9])	([0.2,0.25], [2, 3])	([0.143,0.167], [3, 5])	([1, 1], [1,1])

Table 9
Comprehensively consistent IVIMPR $\tilde{R}^*$.

	HBIS	SHOUGANG	BAOSTEEL	ANSTEEL
HBIS	([1, 1], [1,1])	([2.15, 2.2462], [0.181,0.222])	([1.2, 1.478], [0.224, 0.346])	([4.368, 6.015], [0.048, 0.074])
SHOUGANG	([0.181,0.222], [2.15, 2.2462])	([1, 1], [1,1])	([0.558, 0.6], [1.241, 1.554])	([2.036, 2.443], [0.264,0.332])
BAOSTEEL	([0.224, 0.346], [1.2, 1.478])	([1.241, 1.554], [0.558, 0.6])	([1, 1], [1,1])	([3.649, 4.07], [0.213,0.214])
ANSTEEL	([0.048, 0.074], [4.368, 6.015])	([0.264,0.332], [2.036, 2.443])	([0.213,0.214], [3.649, 4.07])	([1, 1], [1,1])

study new decision-making methods with interval-valued intuitionistic multiplicative information, such as methods based on aggregation operators, similarities, distance measures, and entropy. Furthermore, we shall research the application of IVIMVs in some other fields, including ERP system selection, engineering project management, digital image processing, medical recommendation, software quality assurance management, and teaching quality assessment. Moreover, consensus is a necessary step for GDM, and we will continue to study the consensus for GDM with IVIMPRs following the work of Herrera-Viedma et al. (2014).

CRediT authorship contribution statement

Fanyong Meng: Conceptualization, Methodology, Writing - original draft, Writing - review & editing, Visualization, Project administration, Funding acquisition. **Jie Tang:** Conceptualization, Methodology, Data curation, Writing - original draft, Visualization, Project administration. **Francisco Javier Cabrerizo:** Formal analysis, Resources, Writing - review & editing. **Enrique Herrera-Viedma:** Methodology, Writing - review & editing, Visualization, Project administration, Funding acquisition.

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