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Theorem Verification of the Quantifier-Guided Dominance Degree with the Mean Operator for Additive Preference Relations

José Ramón Trillo [1,*], Francisco Javier Cabrerizo [1], Francisco Chiclana [2], María Ángeles Martínez [3], Francisco Mata [4] and Enrique Herrera-Viedma [1]

1 Department of Computer Science and Artificial Intelligence, Andalusian Research Institute in Data Science and Computational Intelligence, DaSCI, University of Granada, 18071 Granada, Spain; cabrerizo@decsai.ugr.es (F.J.C.); viedma@decsai.ugr.es (E.H.-V.)
2 Institute of Artificial Intelligence, Faculty of Computing, Engineering and Media, De Montfort University, Leicester LE1 9BH, UK; chiclana@dmu.ac.uk
3 Department of Social Work and Social Services, University of Granada, 18001 Granada, Spain; mundodesilencio@ugr.es
4 Department of Computer Science, Andalusian Research Institute in Data Science and Computational Intelligence, DaSCI, University of Jaén, 23071 Jaén, Spain; fmata@ujaen.es
* Correspondence: jrtrillo@ugr.es

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Abstract: Deciding which film is the best or which portfolio is the best for investment are examples of decisions made by people every day. Decision-making systems aim to help people make such choices. In general, a decision-making system processes and analyses the available information to arrive at the best alternative solution of the problem of interest. In the preference modelling framework, decision-making systems select the best alternative(s) by maximising a score or choice function defined by the decision makers' expressed preferences on the set of feasible alternatives. Nevertheless, decision-making systems may have logical errors that cannot be appreciated by developers. The main contribution of this paper is the provision of a verification theorem of the score function based on the quantifier-guided dominance degree (QGDD) with the mean operator in the context of additive preference relations. The provided theorem has several benefits because it can be applied to verify that the result obtained is correct and that there are no problems in the programming of the corresponding decision-making systems, thus improving their reliability. Moreover, this theorem acts on different parts of such systems, since not only does the theorem verify that the order of alternatives is correct, but it also verifies that the creation of the global preference relation is correct.

Keywords: quantifier-guided dominance degree; verification; additive matrix; decision-making system; selection process

MSC: 90B50; 62C86

1. Introduction

From which mobile phone to buy to which road map a company should follow, decisions are part of people's everyday activities. We make some decisions without difficulty, though others are difficult and are the reason why decision-making systems exist [1]. There have been many different systems developed to help experts make decisions based on their opinions or preferences [2–4]. In any case, systems can fail, and there can be errors, called logical errors, which do not affect the functioning of the system but do affect the final result [5], and that may have enormous negative repercussions in financial terms because the systems are supposed to provide good support to decision makers.

In order to avoid logical errors in decision-making systems, specifically in group decision-making (GDM) systems [6–9], it is necessary to apply theories that can be demonstrated mathematically. Consequently, reliable programs must be obtained and supported by mathematical theorems proving their correct functioning.

A theorem is a proposition that can be proved mathematically and lacks a contradictory case that would allow its invalidation [10]. Every mathematical theorem starts from an assumption, called a hypothesis, and a rationale, called a thesis [11]. Consequently, this study proposes a mathematical theorem to verify that the GDM system works correctly when it selects the best alternative(s) by maximising the quantifier-guided dominance degree (QGDD) with the mean operator [12,13].

The QGDD is a concept used to define a score or choice function based on the information provided by the decision makers (experts or users) that is used to provide an ordering of the alternatives, which is performed at the last step of a GDM system [14,15]. Since the application of a choice function is performed at the last step of the GDM process, the system must have worked correctly up to that point. Otherwise, in case of any logical error, the QGDD could provide an incorrect ordering of the alternatives, and the wrong solution to the problem would be obtained. Nevertheless, the theorem provided in this study will allow both experts and researchers to mathematically verify that the result provided by the GDM system is reliable [16].

Logical errors have been analysed in the existent literature [17]. For example, in [18,19], the authors focused on the logical errors produced by the experts but not on the errors that the developed system may have been exhibiting, which may have led to incorrect ranking of the alternatives. In contrast, this article focuses on the logical errors produced by the formal processes implemented by developers in the system but not on the logical errors produced by the experts. For this reason, an advantage of this theorem is that it effectively verifies that the program obtains consistent results and that the programmers have not made mistakes. The theorem also has limitations on its applicability, since it can only be applied when its supporting hypothesis is verified. Consequently, although it can be applied to any area that uses a decision-making system, it is also restricted to the set of systems that use additive preference relations.

2. Preliminaries

2.1. Group Decision-Making System

A GDM system is characterised by a set of experts ($E = \{e_q;\, q = 1, \ldots, b\};\, b \in \mathbb{N}$) who aim to jointly choose the best alternative(s) from a previously defined set of feasible alternatives ($A = \{A_t;\, t = 1, \ldots, n\};\, n \in \mathbb{N}\backslash\{1\}$). This article deals with systems that aim to create an ordering of the alternatives from best to worst, from which the best alternative solution(s) to the problem is chosen [20,21]. The final ranking of the alternatives is derived from the evaluations of the set of alternatives that each expert involved in the GDM problem provides [22,23]. This article is restricted to GDM problems with assessments of pairs of alternatives, which is known as the preferences of one alternative over another [24]. In order to carry out a solution to the problem, this study applies a fuzzy set theory methodology since it assumes that the preference degree of one alternative over another is not restricted to a binary set ($\{0, 1\}$) but a continuum set ($[0, 1]$ or $[0, U]$, $U \in \mathbb{N}$) [25].

A GDM system consists of a series of steps or processes: discussion between the different experts involved to identify the set of alternatives and preference representation format, experts' provision of their evaluations of the alternatives, and obtaining a ranking of the alternatives, which is the solution provided by the system for the problem [13]. In the following, each step is described in more detail (see Figure 1):

- Conducting the discussion among the experts: The problem is analysed by the experts, a feasible set of alternatives is established, and a preference representation structure is agreed upon for use [26].
- Providing assessments of the alternatives: Using the agreed preference representation structure, the experts provide their corresponding preferences (preference relation) for the set of feasible alternatives [24].

- Analysing consensus: This step is optional in a GDM process, but it is often used to ensure that the decision being made is agreed upon by the experts [27,28]. To accomplish this, a group consensus value is defined and computed [29]. If the group consensus is not below a previously established threshold value, denoted by $\alpha \in \mathbb{R}_0^+$, then the system continues processing the experts' preferences to derive a ranking of the alternatives; otherwise, a feedback mechanism is activated to support the experts in reaching the consensus threshold [30].
- Creating the collective preference relation: The experts' preference relations are aggregated into a single collective preference relation [31,32] of $G = (g_{tk};\ t \neq k;\ t, k = 1, \ldots, n)$, with $g_{tk} \in \mathbb{R}$ being the preference of alternative A_t over alternative A_k for the group of experts. Many aggregation operators could be used to derive the collective preference relation, that being the weighed average (WA) operator usually implemented in GDM systems [33], with each expert being associated with a corresponding weighting or importance.
- Computing the ranking of alternatives: Using the collective preference relation, a score or choice function is defined to produce the final (consensus) ranking of the alternatives, which is a solution to the problem offered by the GDM system [34]. This study assumes that the QGDD is the score or choice function [12].

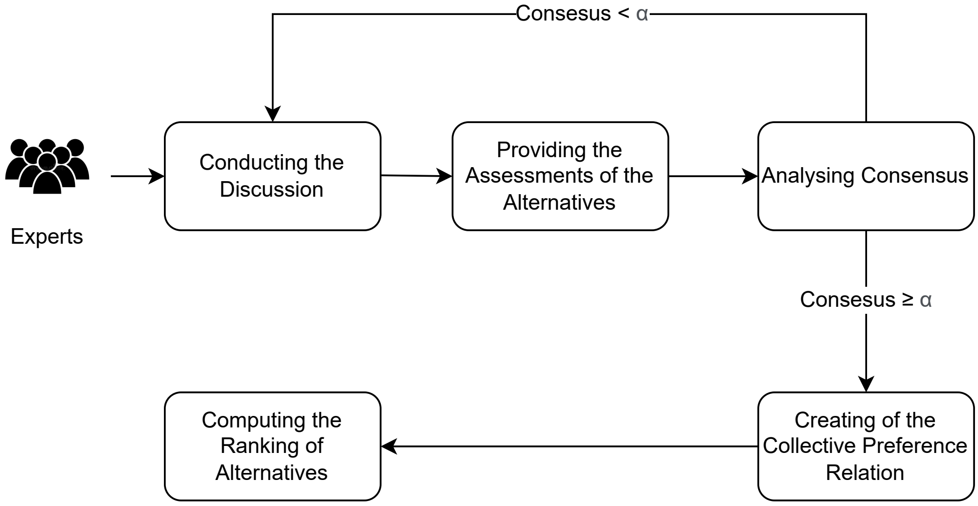

Figure 1. Basic diagram of a GDM system.

2.2. The Quantifier-Guided Dominance Degree

Orlovsky proposed the concept of the dominance degree of one alternative over the rest of the alternatives (all other alternatives) [35] by modelling the concept "all" with the application of the fuzzy set theory-based minimum operator. This approach was later generalized with the OWA operator [36], which has the minimum operators as a particular case. The range of OWA operators is extensive. One of them is the usual average or mean operator, which treats with the same importance all the alternatives that are compared [37]. This is the operator used in this study.

The QGDD with the mean operator computes the average dominance of an alternative over the rest of the alternatives. Using the collective preference relation G, the QGDD with the mean operator of the alternatives A_t $(t = 1, \ldots, n)$ is

$$QGDD_{A_t} = \frac{1}{n-1} \sum_{k=1;k\neq t}^{n} g_{tk}, \tag{1}$$

with n being the number of alternatives and g_{kt} of row k and column t belonging to the matrix G.

From Equation (1), the ordering of the alternatives is produced from a higher dominance degree to a lower dominance degree. The solution alternative(s) of the GDM problem is therefore defined as follows:

$$X^{QGDD} = \{A_s \in A \mid QGDD_{A_s} = \sup_{A_t \in A} QGDD_{A_t}\}, \tag{2}$$

with A being the set of alternatives.

3. Theorem Verification of the Quantifier-Guided Dominance Degree with the Mean Operator

As can be seen in Figure 2, the hypothesis of the theorem affects the second step of the GDM process, since the experts' opinions are to be modelled as additive preference matrices while the thesis of the theorem affects the last two steps of a GDM process, in which the collective preference relation and the ranking of alternatives are calculated. Consequently, the proof of the theorem will verify that the proposed GDM process is correct.

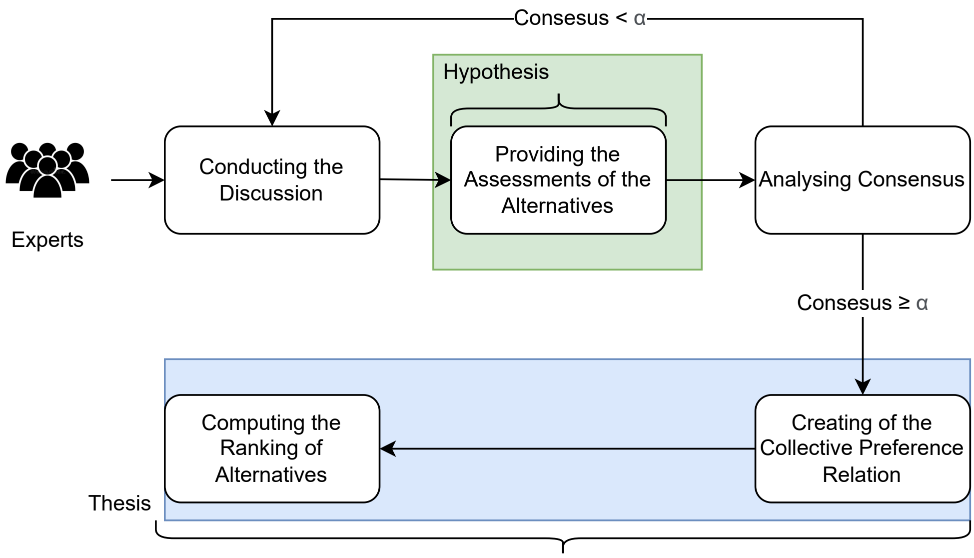

Figure 2. Sections affected by the hypothesis and thesis within the basic outline of a GDM system.

The two definitions below are necessary for understanding of the theorem and its proof:

Definition 1 (Triangular number)**.** *The following sum*

$$\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2};\ n \in \mathbb{N} \tag{3}$$

is called a triangular number [38].

Geometrically, a triangular number coincides with the number of points of the triangular arrangement with n points (shown in Figure 3).

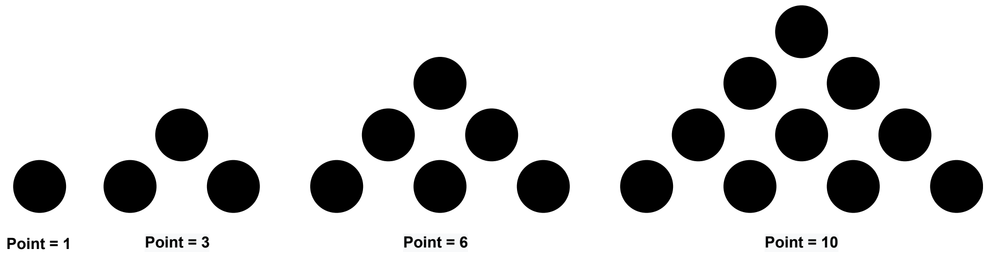

Figure 3. Examples of the triangular number.

Definition 2 (Additive matrix)**.** *A matrix $M = (m_{tk};\ t,k = 1,\ldots,n)$, where m_{tk} is the element of row t and column k belonging to the matrix M of dimensions $n \times n$ with $n \in \mathbb{N}\backslash\{1\}$, is called additive when the main diagonal element is empty and the rest of elements satisfy the following:*

$$m^q_{tk} + m^q_{kt} = U;\ t \neq k;\ t,k = 1,\ldots,n;\ U \in \mathbb{N}. \tag{4}$$

With these two definitions in place, we can proceed to state the verification theorem proposed in this study:

Theorem 1. *For a decision-making process, let $E = \{e_1,\ldots,e_b\}$, where $b \in \mathbb{N}$, be the set of experts with non-negative weights $W = \{w_{e_1},\ldots,w_{e_b}\}$, verifying $\sum_{q=1}^{b} w_{e_q} = 1$. Let $A = \{A_1,\ldots,A_n\}$, where $n \in \mathbb{N}\backslash\{1\}$, be the set of alternatives. Let $\{M_{e_q} = \left(m^q_{tk}\right), q = 1,\ldots,b;\ t,k = 1,\ldots,n\}$ be the set of additive matrices representing the preferences of the set of experts on the set of alternatives. Then, the following expression is verified:*

$$\sum_{p=1}^{n} QGDD_{A_p} = \frac{U \cdot n}{2}, \tag{5}$$

where $QGDD_{A_p}$ is the QGDD of alternative A_p, n is the number of alternatives, and $U \in \mathbb{N}$.

Proof. For the collective preference matrix $G = (g_{tk};\ t,k = 1,\ldots,n)$ where

$$g_{tk} = \sum_{q=1}^{b} w_{e_q} \cdot m^q_{tk};\ w_{e_q} \in W,\ m^q_{tk} \in M_{e_q} \tag{6}$$

is an additive matrix, let $t \neq k$. Since $\{M_{e_q};\ q = 1,\ldots,b\}$ are additive matrices, it holds that

$$g_{kt} = \sum_{q=1}^{b} w_{e_q} \cdot m^q_{kt} = \sum_{q=1}^{b} w_{e_q} \cdot (U - m^q_{tk}) = U - \sum_{q=1}^{b} w_{e_q} \cdot m^q_{tk}.$$

Therefore, we conclude that

$$g_{tk} + g_{kt} = \sum_{q=1}^{b} w_{e_q} \cdot m^q_{tk} + U - \sum_{q=1}^{b} w_{e_q} \cdot m^q_{tk} = U.$$

The mean QGDD of alternative A_p is

$$QGDD_{A_p} = \frac{\sum\limits_{j=1;p\neq j}^{n} g_{pj}}{n-1}. \tag{7}$$

Therefore, it holds that

$$\sum_{p=1}^{n} QGDD_{A_p} = \frac{\sum\limits_{p=1}^{n}\sum\limits_{j=1;p\neq j}^{n} g_{pj}}{n-1}.$$

Since $g_{pj} + g_{jp} = U$, the following algebraic manipulation yields

$$\frac{\sum\limits_{p=1}^{n}\sum\limits_{j=1;p\neq j}^{n} g_{pj}}{n-1} = \frac{\sum\limits_{p=1}^{n}\sum\limits_{j=p+1}^{n} [g_{pj}+g_{jp}]}{n-1} = \frac{\sum\limits_{p=1}^{n}\sum\limits_{j=p+1}^{n} U}{n-1} = \frac{U\cdot\sum\limits_{p=1}^{n}\sum\limits_{j=p+1}^{n} 1}{n-1} = \frac{U\cdot\sum\limits_{p=1}^{n}(n-p)}{n-1}$$
$$= \frac{U\cdot\sum\limits_{r=0}^{n-1} r}{n-1} = \frac{U\cdot(n-1)\cdot(n+1-1)}{2\cdot(n-1)} = \frac{U\cdot n}{2}.$$

Summarising, it is proven that

$$\sum_{p=1}^{n} QGDD_{A_p} = \frac{U\cdot n}{2},$$

when n is the number of alternatives. □

4. Illustrative Example

In this section, we present a GDM problem where four experts ($E = \{e_1, e_2, e_3, e_4\}$) have to choose where to invest money to make improvements in a city. They have four options: to improve (A_1) gardens, to improve pavement (A_2), to improve healthcare (A_3), and to renovate old buildings (A_4).

The experts start debating, and once the debate is over, they express their preferences by using numeric values in the $[0,1]$ interval (i.e., by using fuzzy preference relations) [35,39]. Concretely, the fuzzy preference relations given by each expert are as follows:

$$M_{e_1} = \begin{bmatrix} - & 0.5 & 0.0 & 0.5 \\ 0.5 & - & 0.1 & 0.5 \\ 1.0 & 0.9 & - & 0.8 \\ 0.5 & 0.5 & 0.2 & - \end{bmatrix} \quad M_{e_2} = \begin{bmatrix} - & 0.4 & 0.1 & 0.8 \\ 0.6 & - & 0.1 & 0.5 \\ 0.9 & 0.9 & - & 0.9 \\ 0.2 & 0.5 & 0.1 & - \end{bmatrix}$$

$$M_{e_3} = \begin{bmatrix} - & 0.5 & 0.2 & 0.7 \\ 0.5 & - & 0.2 & 0.9 \\ 0.8 & 0.8 & - & 0.9 \\ 0.3 & 0.1 & 0.1 & - \end{bmatrix} \quad M_{e_4} = \begin{bmatrix} - & 0.4 & 0.1 & 0.4 \\ 0.6 & - & 0.0 & 0.5 \\ 0.9 & 1.0 & - & 1.0 \\ 0.6 & 0.5 & 0.0 & - \end{bmatrix}$$

By applying the theorem developed in Section 3, we proceed to see that the hypothesis is satisfied because all preference relations are additive matrices. Therefore, since all matrices satisfy that hypothesis, the verification theorem can be applied.

Once the experts have provided their preference relations, the consensus achieved should be calculated. However, in this example, as our objective is to show the applicability of the proposed theorem, we suppose that the group consensus is not below the consensus threshold value, and we proceed to the following step (i.e., the calculation of the collective preference relation).

To calculate the collective preference relation, each expert has to be assigned a weight. In this illustrative example, the WA operator will be used, and all experts will have the same weight (i.e., $w_{e_q} = 0.25; q = 1, \ldots, b$). The following collective preference relation

$$G = \begin{bmatrix} - & 0.45 & 0.1 & 0.6 \\ 0.55 & - & 0.1 & 0.6 \\ 0.9 & 0.9 & - & \mathbf{0.9} \\ 0.4 & 0.4 & \mathbf{0.2} & - \end{bmatrix}$$

would be detected to be erroneous by the verification theorem since it is not an additive matrix, and it would be corrected to become

$$G = \begin{bmatrix} - & 0.45 & 0.1 & 0.6 \\ 0.55 & - & 0.1 & 0.6 \\ 0.9 & 0.9 & - & \mathbf{0.9} \\ 0.4 & 0.4 & \mathbf{0.1} & - \end{bmatrix}.$$

Thus, the mean QGDDs of the alternatives from G are obtained (see Table 1).

Table 1. Mean QGDD of each of the alternatives.

Values or Alternatives	A_1	A_2	A_3	A_4
QGDD	0.38	0.42	0.9	0.3

To verify that the results obtained are correct, it is necessary to show that the thesis of the verification theorem is verified:

$$0.38 + 0.42 + 0.9 + 0.3 = 2 = \frac{1 \times 4}{2} = 2$$

Consequently, we obtain that the order of preference of the alternatives is $A_3 \succ A_2 \succ A_1 \succ A_4$. Therefore, option A_3 (i.e., the investment to improve healthcare) is the one chosen by the group of experts.

5. Discussion and Conclusions

The theorem proposed in this study shows that GDM systems can be verified in a simple and fast way. This verification allows programmers to know if the system is well-implemented or if there is a logical error. In addition, experts can also verify that the solution provided by the system is reliable and that the decision is well-founded based on their evaluations. Concretely, this theorem has the following advantages:

- Verification of errors: With this demonstration, it is possible to verify that the decision-making process has been carried out correctly and that there are no errors when applied to a concrete decision-making problem. This is helpful when performing a programming process [40].
- It can be applied without assigning weights to the experts by using the arithmetic mean, particularly in the case of the OWA operator.
- Applicable to several areas of study related to the decision-making process: This demonstration can be applied to different areas of decision-making systems, from GDM processes to large-scale group decision-making processes [41], including processes conducted in multi-granular [21] and multi-criteria environments [26].
- Mathematical proof: Being a proven theorem and not having any counterexample invalidating it, this allows researchers to use the theorem to base their obtained results on a proven mathematical theory. This implies a greater solidity in the conducted study.

These advantages are restricted to the fulfilment of the hypothesis of the developed theorem. For this reason, the main limitation of the application of the theorem is its own hypothesis. Nonetheless, if the hypothesis is fulfilled by the system, then it can be applied to the system, regardless of the area which it belongs to. This shows a practical advantage because it is not subject to a specific area.

Regarding the practical relevance of the proposed solution, it can be mentioned that all works using a decision-making system with additive matrices and using the QGDD for ordering of the alternatives should use this theorem to reinforce that the order established in the QGDD is reliable and that there are no variations that favour one alternative over another. Furthermore, it can be seen how the application of a mathematical theorem to a decision-making method improves the reliability of that method. This improved reliability means that the users who have participated in the decision-making process can feel more confident with the decision made by the system.

In addition, the generality of application of this theorem is also its major limitation because the theorem is restricted to the fulfilment of the hypothesis, which means that it cannot be applied if any of the matrices used is not an additive matrix. For future work, two strands can be analysed. The first is to investigate new theorems for other score or choice functions such as the quantifier-guided non-dominance degrees (QGNDDs), and the second is to investigate a decision-making method that transforms non-additive matrices into additive matrices.

Finally, as a conclusion, this paper can show how mathematical theorems are useful to reinforce the results shown by decision-making systems, making them more robust and reliable.

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References

1. Power, D.J. *Decision Support Systems: Concepts and Resources for Managers*; Quorum Books: Westport, CN, USA, 2002.
2. Mardani, A.; Jusoh, A.; Zavadskas, E.K. Fuzzy multiple criteria decision-making techniques and applications—Two decades review from 1994 to 2014. *Expert Syst. Appl.* **2015**, *42*, 4126–4148.
3. Wang, L.; Wang, H. An integrated qualitative group decision-making method for assessing health-care waste treatment technologies based on linguistic terms with weakened hedges. *Appl. Soft Comput.* **2022**, *117*, 108435.
4. Kaklauskas, A.; Dzitac, D.; Sliogeriene, J.; Lepkova, N.; Vetloviene, I. VINERS method for the multiple criteria analysis and neuromarketing of best places to live. *Int. J. Comput. Commun. Control* **2019**, *14*, 629–646.
5. Thorat, S.A.; Kshirsagar, D.P. Developing logic building, problem solving, and debugging programming skills among students. *J. Eng. Educ. Transform.* **2019**, *34*, 402–406.
6. Bhattacharyya, S.; Valeriani, D.; Cinel, C.; Citi, L.; Poli, R. Anytime collaborative brain-computer interfaces for enhancing perceptual group decision making. *Sci. Rep.* **2021**, *11*, 17008.
7. Cabrerizo, F.J.; Ureña, R.; Pedrycz, W.; Herrera-Viedma, E. Building consensus in group decision making with an allocation of information granularity. *Fuzzy Sets Syst.* **2014**, *255*, 115–127.
8. Cabrerizo, F.J.; Al-Hmouz, R.; Morfeq, A.; Balamash, A.S.; Martínez, M.A.; Herrera-Viedma, E. Soft consensus measures in group decision making using unbalanced fuzzy linguistic information. *Soft Comput.* **2017**, *21*, 3037–3050.

9. Liu, S.; Zhang, J.; Niu, B.; Liu, L.; He, X. A novel hybrid multi-criteria group decision-making approach with intuitionistic fuzzy sets to design reverse supply chains for COVID-19 medical waste recycling channels. *Comput. Ind. Eng.* **2022**, *169*, 108228.
10. Moura, L.D.; Ullrich, S. The lean 4 theorem prover and programming language. In *Automated Deduction—CADE 28*; Platzer, A., Sutcliffe, G. Eds.; Lecture Notes in Computer Science, 12699; Springer: Cham, Switzerland, 2021; pp. 625–635.
11. Kovács, Z.; Recio, T.; Tabera, L.F.; Vélez, M.P. Dealing with degeneracies in automated theorem proving in geometry. *Mathematics* **2021**, *9*, 1964.
12. Cabrerizo, F.J.; Morente-Molinera, J.A.; Pedrycz, W.; Taghavi, A.; Herrera-Viedma, E. Granulating linguistic information in decision making under consensus and consistency. *Expert Syst. Appl.* **2018**, *99*, 83–92.
13. Chiclana, F.; Herrera, F.; Herrera-Viedma, E. Integrating multiplicative preference relations in a multipurpose decision making model based on fuzzy preference relations. *Fuzzy Sets Syst.* **2001**, *122*, 277–291.
14. Yuan, Y.; Cheng, D.; Zhou, Z. A minimum adjustment consensus framework with compromise limits for social network group decision making under incomplete information. *Inf. Sci.* **2021**, *549*, 249–268.
15. Roubens, M. Fuzzy sets and decision analysis. *Fuzzy Sets Syst.* **1997**, *90*, 199–206.
16. Liu, Z.; He, X.; Deng, Y. Network-based evidential three-way theoretic model for large-scale group decision analysis. *Inf. Sci.* **2021**, *547*, 689–709.
17. Sałabun, W.; Shekhovtsov, A.; Kizielewicz, B. A new consistency coefficient in the multi-criteria decision analysis domain. In *Computational Science – ICCS 2021*; Paszynski, M., Kranzlmüller, D., Krzhizhanovskaya, V.V., Dongarra, J.J., Sloot, P.M.A., Eds.; Lecture Notes in Computer Science, 12742; Springer, Cham, 2021; pp. 715–727.
18. Sodenkamp, M.A.; Tavana, M.; Di Caprio, D. An aggregation method for solving group multi-criteria decision-making problems with single-valued neutrosophic sets. *Appl. Soft Comput.* **2018**, *71*, 715–727.
19. Magnani, L. Naturalizing logic: Errors of reasoning vindicated: Logic reapproaches cognitive science. *J. Appl. Log.* **2015**, *13*, 13–36.
20. Hwang, C.L.; Lin, M.J. *Group Decision Making Under Multiple Criteria: Methods and Applications*; Springer: Berlin/Heidelberg, Germany, 1987.
21. Morente-Molinera, J.A.; Kou, G.; Pang, C.; Cabrerizo, F.J.; Herrera-Viedma, E. An automatic procedure to create fuzzy ontologies from users' opinions using sentiment analysis procedures and multi-granular fuzzy linguistic modelling methods. *Inf. Sci.* **2019**, *476*, 222–238.
22. Liu, F.; Huang, M.; Pedrycz, W.; Zhao, H. Group decision making based on flexibility degree of fuzzy numbers under a confidence level. *IEEE Trans. Fuzzy Syst.* **2021**, *29*, 1640–1653.
23. Miebs, G.; Kadziński, M. Heuristic algorithms for aggregation of incomplete rankings in multiple criteria group decision making. *Inf. Sci.* **2021**, *560*, 107–136.
24. Herrera-Viedma, E.; Palomares, I.; Li, C.C.; Cabrerizo, F.J.; Dong, Y.C.; Chiclana, F.; Herrera F. Revisiting fuzzy and linguistic decision making: Scenarios and challenges for making wiser decisions in a better way. *IEEE Trans. Syst. Man, Cybern. Syst.* **2021**, *51*, 191–208.
25. Bustince, H.; Barrenechea, E.; Pagola, M.; Fernández, J.; Xu, Z.S.; Bedregal, B.; Montero, J.; Hagras, H.; Herrera, F.; De Baets, B. A historical account of types of fuzzy sets and their relationships. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 179–194.
26. Morente-Molinera, J.A.; Kou, G.; Samuylov, K.; Cabrerizo, F.J.; Herrera-Viedma, E. Using argumentation in expert's debate to analyze Multi-criteria group decision making method results. *Inf. Sci.* **2021**, *573*, 433–452.
27. Cabrerizo, F.J.; Chiclana, F.; Al-Hmouz, R.; Morfeq, A.; Balamash, A.S.; Herrera-Viedma, E. Fuzzy decision making and consensus: Challenges. *J. Intell. Fuzzy Syst.* **2015**, *29*, 1109–1118.
28. Chen, X.; Ding, Z.; Dong, Y. Adjustments in group decision making with opinions evolution. *IEEE Trans. Syst. Man, Cybern. Syst.* **2021**, *51*, 2299–2311.
29. Zhang, Y.; Chen, X.; Gao, L.; Dong, Y.; Pedrycz, W. Consensus reaching with trust evolution in social network group decision making. *Expert Syst. Appl.* **2022**, *188*, 116022.
30. Zhang, Z.; Li, Z.; Gao, Y. Consensus reaching for group decision making with multi-granular unbalanced linguistic information: A bounded confidence and minimum adjustment-based approach. *Inf. Fusion* **2021**, *74*, 96–110.
31. Blanco-Mesa, F.; León-Castro, E.; Merigó, J.M. A bibliometric analysis of aggregation operators. *Appl. Soft Comput.* **2019**, *81*, 105488.
32. Wu, Z.; Yang, X.; Xu, J. Dual models and return allocations for consensus building under weighted average operators. *IEEE Trans. Syst. Man, Cybern. Syst.* **2021**, *51*, 7164–7176.
33. Wan, B., Lu, R.; Han, M. Weighted average LINMAP group decision making method based on q-rung orthopair triangular fuzzy numbers. *Granul. Comput.* **2021**, *7*, 489–503.
34. Kacprzak, D. A novel extension of the tecnique for order preference by similarity to ideal solution method with objective criteria weights for group decision making with interval numbers. *Entropy* **2021**, *23*, 1460.
35. Orlovsky, S. Decision-making with a fuzzy preference relation. *Fuzzy Sets Syst.* **1978**, *1*, 155–167.
36. Yager R.R. On ordered weighted averaging aggregation operators in Multi-criteria decision-making. *IEEE Trans. Syst. Man, Cybern.* **1988**, *18*, 183–190.
37. Yager, R.R. Families of OWA operators. *Fuzzy Sets Syst.* **1993**, *59*, 125–148.
38. Tattersall, J.J. *Elementary Number Theory in Nine Chapters*; Cambridge University Press: Cambridge, UK, 1999.

39. Xu, Y.; Li, M.; Cabrerizo, F.J.; Chiclana, F.; Herrera-Viedma, E. Algorithms to detect and rectify multiplicative and ordinal inconsistencies of fuzzy preference relations in *IEEE Trans. Syst. Man, Cybern. Syst.* **2021**, *51*, 3498–3511.
40. Sawik, T.; Sawik, B. A rough cut cybersecurity investment using portfolio of security controls with maximum cybersecurity value. *Int. J. Prod. Res.*, **2021**, 1–17. https://doi.org/10.1080/00207543.2021.1994166.
41. Palomares, I. *Large Group Decision Making: Creating Decision Support Approaches at Scale*; Springer: Cham, Switzerland, 2018.