A General Procedure to Estimate Missing Values for Incomplete Fuzzy Preference Relations

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Abstract

In this paper we present a general procedure to complete fuzzy preference relations with missing values. Based on a consistency principle characterized by the transitivity property, this procedure estimate those missing values of an incomplete fuzzy preference relation. Different properties have been proposed to model the concept of transitivity. For each one of these, we present a corresponding measure of consistency of the opinions provided by an expert. **Keywords:** Fuzzy Preference Relations, Missing Information, Consistency, Transitivity

1 Introduction

Consistency and *lack of information* are serious challenges when dealing with Group Decision Making problems. The first one refers to the capability of the experts to express their preferences in a consistent way, that is, without contradiction. The second one appears when experts are not able to properly give all the information that they are asked for. There may be many different motives why an expert could not be able to give some of his or her preference opinions about the alternatives on a problem. For example, the expert may not have a precise or sufficient level of knowledge about some of the alternatives; the expert may not able to discriminate the degree to which some options are better than others; or maybe there are too many alternatives and the expert cannot give his/her preferences in a consistent manner. In those situations, an expert may have/want to give incomplete information.

To deal with consistency problems it is important to properly characterize what consistency properties the preferences should comply with [2, 5]. Also, it would be desirable to have a *measurement* of consistency of the opinion expressed by the experts. This would enable us to develop models [6] to solve decision making problems in which consistent information would be considered more valuable that inconsistent information. In [1, 6] we defined and used some consistency measures to develop complete models for group decision making problems that obtains solutions by giving different importance weights according to the level of consistency of the preferences. In these models the consistency measures were defined using the the additive transitivity property [11].

In the literature we can find different approaches to deal with lack of information [1, 6, 8, 9, 13]. In particular, in [1] we developed a procedure which was able to estimate missing preference values in incomplete fuzzy preference relations. This procedure was also based on the additive transitivity property.

In this paper we generalize our procedure to estimate missing values in incomplete fuzzy preference relations. This general procedure is also guided by consistency, but with the main difference of being able to be characterized by using any of the different transitivity

properties proposed in the literature. In such a way, we can freely choose the transitivity property that the preference relations should comply with, and present a more flexible estimation procedure [1, 5, 6].

To do so, the paper is set out as follows: In Section 2 we present our preliminaries. In section 3 we define consistency measures for incomplete fuzzy preference relations, each one of them associated to one of the known different transitivity properties. In section 4 we present the general estimation procedure for incomplete fuzzy preference relations. Finally, in section 5 we point out our conclusions.

2 Preliminaries

In this section we present the concepts of *complete* and *incomplete fuzzy preference relations*, as well as those of *transitivity*, *consistency* and *completeness* for fuzzy preference relations that will be needed throughout the rest of the paper.

2.1 Fuzzy Preference Relations

In *Group Decision Making*, experts have to express their preferences about a set of given alternatives $X = \{x_1, ..., x_n\}$, $(n \geq 2)$ in order to find the best of those alternatives. There exist several different formats which can be used to represent experts' preferences, with *Fuzzy Preference Relations* being one the most widely used in the literature.

Definition 1 ([7, 10]) *A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, i.e., it is characterized by a membership function*

$$\mu_P : X \times X \longrightarrow [0, 1]$$

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ik})$, being $p_{ik} = \mu_P(x_i, x_k)$ $(\forall i, k \in \{1, \ldots, n\})$ interpreted as the preference degree or intensity of the alternative x_i over x_k: $p_{ik} = 1/2$ indicates indifference between x_i and x_k $(x_i \sim x_k)$, $p_{ik} = 1$ indicates that x_i is absolutely preferred to x_k, and $p_{ik} > 1/2$ indicates that x_i is preferred to x_k $(x_i \succ x_k)$. Based on this interpretation we have that $p_{ii} = 1/2 \ \forall i \in \{1, \ldots, n\}$ $(x_i \sim x_i)$.

Although fuzzy preference relations are very expressive and easy to use, and despite of the fact that individual fuzzy preference relations can be easily aggregated into group preferences [4, 5, 7, 11, 12], they also present some drawbacks. One of them refers to the problems of *lack of information*. It is not unusual to find that some experts could have difficulties in expressing every preference degree between every pair of alternatives. These difficulties appear due to different reasons: the expert does not have a precise or sufficient level of knowledge about some of the alternatives, the expert is not able to discriminate the degree to which some options are better than others or maybe there are too many alternatives and the expert cannot give every preference degree in a consistent manner. In these situations the experts may choose not to provide every preference degree that they are required to, and thus, we have to deal with *incomplete fuzzy preference relations*:

Definition 2 *A function $f : X \longrightarrow Y$ is* partial *when not every element in the set X necessarily maps onto an element in the set Y. When every element from the set X maps onto one element of the set Y then we have a* total *function.*

Definition 3 ([6]) *An Incomplete Fuzzy Preference Relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$ that is characterized by a partial membership function.*

When an expert does not provide a particular p_{ik} we will call it a *missing value* and we will represented it as $p_{ik} = x$. We also introduce the following sets:

$$A = \{(i, k) \mid i, k \in \{1, \ldots, n\} \wedge i \neq k\}$$

$$MV = \{(i, k) \in A \mid p_{ik} = x\}$$

$$EV = A \setminus MV$$

$$EV_i = \{(i, k), (k, i) \in EV\}$$

where MV is the set of pairs of alternatives for which the preference degree of the first alternative over the second one is a missing value; EV is the set of pairs of alternatives for which the expert provides preference values; and EV_i is the set of pairs of alternatives involving alternative x_i for which the expert provides preference values. We do not take into account the preference value of one alternative over itself as this is always assumed to be equal to 0.5.

2.2 Transitivity, Consistency and Completeness Concepts

We define the *consistency* of a fuzzy preference relation as a degree to which the information on the relation is not contradictory. Because the preference degrees expressed in a preference relation can be freely chosen by the experts, we cannot directly assume that they comply with any particular consistency property. However, it is obvious that an inconsistent source of information should not be considered as useful as a consistent one. Therefore, to study the consistency of the preference relations to correctly solve decision problems we may face is quite important.

Consistency is usually characterized by *transitivity*, which represents the idea that the preference value obtained by directly comparing two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives [3], i.e., $x_i \succeq x_{j_1} \succeq ... \succeq x_{j_r} \succeq x_k \Rightarrow x_i \succeq x_k$.

In the literature, different properties to model the concept of transitivity have been suggested, as for example [2]:

- *Triangle Condition*
  $p_{ij} + p_{jk} \geq p_{ik} \quad \forall i, j, k,$
- *Weak Transitivity*:
  $min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq 0.5 \quad \forall i, j, k,$
- *Max-Min Transitivity*:
  $p_{ik} \geq min\{p_{ij}, p_{jk}\} \quad \forall i, j, k,$
- *Max-Max Transitivity*:
  $p_{ik} \geq max\{p_{ij}, p_{jk}\} \quad \forall i, j, k,$
- *Restricted Max-Min Transitivity*:
  $min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow$
  $p_{ik} \geq min\{p_{ij}, p_{jk}\} \quad \forall i, j, k,$
- *Restricted Max-Max Transitivity*:
  $min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow$
  $p_{ik} \geq max\{p_{ij}, p_{jk}\} \quad \forall i, j, k,$
- *Additive Transitivity*:
  $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \forall i, j, k$

When for every three options in the problem $x_i, x_j, x_k \in X$ their associated preference degrees p_{ij}, p_{jk}, p_{ik} fulfil one of the previously presented transitivity properties we will consider the preference relation completely consistent. For example, a preference relation will be considered as *additive consistent* if its preference values comply with the additive transitivity property, or *Max-Min consistent* if it complies with the Max-Min transitivity property.

For every pair of alternatives x_i and x_k we define its *completeness measure* as

$$\alpha_{ik} = \frac{\#EV_i + \#EV_k - \#(EV_i \cap EV_k)}{4(n-1) - 2}$$

$\alpha_{ik} = 1$ means that all the possible preference value involving alternatives x_i and x_k are provided by the expert. This value decreases as the number of missing preference values involving those alternatives in the fuzzy preference relation increases. If all the preference value involving both alternatives are missing then $\alpha_{ik} = 0$.

3 Consistency Measures Based on Different Transitivity Properties

In [6] we investigated and developed a complete decision making model which is able to handle incomplete information situations. In that investigation, we made use of the additive transitivity property to define consistency measures of fuzzy preference relations. Because additive transitivity may not be the most appropriate property to model consistency for certain problems (it can be a very restrictive property), in this section we will

generalize our consistency measures to accommodate any of the above transitivity properties.

The transitivity properties presented in 2.2 can be used to test whether a fuzzy preference relation is consistent (according to that transitivity property) or not. However, they cannot directly be used to measure the level of consistency the preference relation is, i.e, given two inconsistent fuzzy preference relations we cannot discern which one is the most inconsistent one.

As the preference relations that we are dealing with could be incomplete, it may be also necessary to rewrite the particular transitivity properties we are studying to be able to check that for every given p_{ik} in the preference relation the property is satisfied or not, and in the latter case, to measure how inconsistent every value is with respect to the rest of information in the relation.

Example 1 *The* additive transitivity *property for a particular preference relation can be rewritten as as:*

$$p_{ik} = p_{ij} + p_{jk} - 0.5 \quad \forall i,j,k \quad (exp.\ 1) \quad (1)$$

and from that expression, and knowing that additive transitivity implies reciprocity ($p_{ik} = 1 - p_{ki}$) we can also deduce that:

$$p_{ik} = p_{jk} - p_{ji} + 0.5 \quad \forall i,j,k \quad (exp.\ 2) \quad (2)$$

and that:

$$p_{ik} = p_{ij} - p_{kj} + 0.5 \quad \forall i,j,k \quad (exp.\ 3) \quad (3)$$

Example 2 Max-Min transitivity *property:*

$$p_{ik} \geq min\{p_{ij}, p_{jk}\} \quad \forall i,j,k \quad (exp.\ 1) \quad (4)$$

cannot be rewritten in any other form.

In order to check whether a particular value p_{ik} given by the expert is consistent or not, preference values relating both alternatives x_i and x_k with other different alternatives are to be known or provided by the expert. The sets of alternatives (x_j) that can be used to check the consistency of a preference value p_{ik} are represented as H^l_{ik} (l is the number of expressions that a particular transitivity property implies):

Example 3 *The H^l_{ik} sets for the additive transitivity property are:*

- *For (exp. 1) (1):*

$$H^1_{ik} = \{j \neq i,k \mid (i,j),(j,k) \in EV\}$$

- *For (exp. 2) (2):*

$$H^2_{ik} = \{j \neq i,k \mid (j,i),(j,k) \in EV\}$$

- *For (exp. 3) (3):*

$$H^3_{ik} = \{j \neq i,k \mid (i,j),(k,j) \in EV\}$$

Example 4 *For Max-Min transitivity property there is only one H^l_{ik} set corresponding to (exp. 1) (4):*

$$H^1_{ik} = \{j \neq i,k \mid (i,j),(j,k) \in EV\}$$

Once that we know the alternatives that can be used to check the consistency of a preference value p_{ik} we define a partial consistency degree according to every expression l as follows:

$$cl^l_{ik} = \begin{cases} \dfrac{\sum_{j \in H^l_{ik}} cl^{jl}_{ik}}{\#H^l_{ik}} & if\ (\#H^l_{ik} \neq 0) \\ 0 & otherwise \end{cases}$$

where cl^{jl}_{ik} is a normalized distance function between the value p_{ik} and the value that would be obtained by applying expression l. Note that if $H^l_{ik} = \emptyset$ then expression l cannot be applied, and we assign $cl^{jl}_{ik} = 0$).

Example 5 *If the additive transitivity property is used we have:*

For (exp. 1) (1):

$$cl^{j1}_{ik} = (2/3) \cdot |p_{ik} - (p_{ij} + p_{jk} - 0.5)|$$

For (exp. 2) (2):

$$cl^{j2}_{ik} = (2/3) \cdot |p_{ik} - (p_{jk} - p_{ji} + 0.5)|$$

For (exp. 3) (3):

$$cl^{j3}_{ik} = (2/3) \cdot |p_{ik} - (p_{ij} - p_{kj} + 0.5)|$$

Example 6 *For Max-Min transitivity property we have:*

For (exp. 1) (4):

$$cl_{ik}^{j1} = \begin{cases} |p_{ik} - mm| & if\ (p_{ik} < mm) \\ 0 & otherwise \end{cases}$$

with $mm = min\{p_{ij}, p_{jk}\}$.

Finally, the consistency level of the preference value p_{ik} is obtained as a combination of the partial consistency degrees and the completeness measure presented in section 2.2:

$$CL_{ik} = \alpha_{ik} \cdot (1 - \phi(cl_{ik}^{l}))$$

where ϕ corresponds to the arithmetic mean.

$CL_{ik} = 1$ means that the preference value p_{ik} is completely consistent with the other information in the preference relation.

4 Generalized Procedure to Estimate Missing Values

In [1, 6] we developed an iterative procedure that allows the estimation of missing values in incomplete fuzzy preference relations by means of the application of the additive transitivity property. In this section we will generalize that procedure to be able to use any of the transitivity properties in the estimation process, and thus to provide a more flexible procedure in terms of its applicability.

In order to develop the procedure two different tasks have to be carried out:

A) Establish the elements that can be estimated in each step of the procedure, and

B) produce the particular expression that will be used to estimate a particular missing value.

A) Elements to be estimated in step h

The subset of missing values MV that can be estimated in step h of our procedure is denoted by EMV_h (*estimated missing values*) and defined as follows:

$$EMV_h = \left\{(i,k) \in RMV_h \mid \exists j \in \{H_{ik}^h\}\right\}$$

where $EMV_0 = \emptyset$ (by definition), RMV_h stands for Remaining Missing Values

$$RMV_h = MV \setminus \bigcup_{l=0}^{h-1} EMV_l$$

and where $H_{ik}^h = \bigcup_{l=0}^{h-1} H_{ik}^l$ where the H_{ik}^l sets are computed in every iteration as in section 3 with the known and estimated values in the relation from the previous iteration.

Example 7 *For the Max-Min transitivity property, in iteration h of the procedure the H_{ik} set is:*

$$H_{ik} = H_{ik}^1 = \{j \mid (i,j),(j,k) \in A \setminus RMV_h\}$$

When $EMV_{maxIter} = \emptyset$ with $maxIter > 0$ the procedure will stop as there will not be any more missing values to be estimated.

B) Expression to estimate a particular value p_{ik} in step h

In order to estimate a particular value p_{ik} with $(i,k) \in EMV_h$, we propose the application of the following function:

function estimate_p(i,k)

1. $\mathcal{K} = 0$
2. for every expression l to evaluate:
3. $cp_{ik}^{l} = 0$
4. if $\#H_{ik}^{l} \neq 0 \Rightarrow cp_{ik}^{l} = \frac{\sum_{j \in H_{ik}^{l}} cp_{ik}^{jl}}{\#H_{ik}^{l}}$; $\mathcal{K}$++
5. end for
6. Calculate $cp_{ik} = \frac{1}{\mathcal{K}} \cdot \sum_{l} cp_{ik}^{l}$

end function

cp_{ik}^{jl} is the *minimum* value that would be obtained by the application of expression l and cp_{ik} is the final estimated value.

Example 8 *For the additive transitivity property we have that:*

$$cp_{ik}^{j1} = p_{ij} + p_{jk} - 0.5,$$
$$cp_{ik}^{j2} = p_{jk} - p_{ji} + 0.5,$$
$$cp_{ik}^{j3} = p_{ij} - p_{kj} + 0.5.$$

Example 9 *For the Max-Min transitivity property we have that:*

$$cp_{ik}^{j1} = min\{p_{ij}, p_{jk}\}.$$

Finally, the *iterative estimation procedure pseudo-code* is as follows:

ESTIMATION PROCEDURE

0. $EMV_0 = \emptyset$
1. $h = 1$
2. while $EMV_h \neq \emptyset$ {
3. for every $(i,k) \in EMV_h$ {
4. estimate_p(i,k)
5. }
6. $h++$
7. }

5 Conclusions

In this paper we have presented a general consistency based procedure which allows the estimation of missing values in incomplete fuzzy preference relations. This general procedure allows the use of different transitivity properties to model the consistency concept to implement in the particular decision making problem to solve. Different consistency measures for the different transitivity properties have been defined. The proposed procedure generalises those presented in [1, 6].

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References

[1] Alonso, S., Chiclana, F., Herrera, F., Herrera-Viedma, E., Alcal-Fdez, J.: A Consistency Based Procedure to Estimate Missing Pairwise Preference Values. *International Journal of Intelligent Systems*, in press (2005)

[2] Chiclana, F., Herrera, F., Herrera-Viedma, E.: Reciprocity and Consistency of Fuzzy Preference Relations. In: De Baets, B., Fodor, J. (Eds.). *Principles of Fuzzy Preference Modelling and Decision Making*, Academia Press, (2003) 123–142

[3] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Application, (Academic Press, New York, 1980).

[4] Fodor, J., Roubens, M.: *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwert Academic Publichers, Dordrecht (1994).

[5] Herrera-Viedma, E., Herrera, F., Chiclana, F., Luque, M.: Some issues on consistency of fuzzy preference relations. *European Journal of Operational Research* **154** (2004) 98–109

[6] Herrera-Viedma, E., Chiclana, F., Herrera, F., Alonso, S.: A Group Decision Making Model with Incomplete Fuzzy Preference Relations Based on Additive Consistency. *Technical Report #SCI2S-2004-11.* University of Granada (2004). *http://sci2s.ugr.es/publications/*

[7] Kacprzyk, J.: Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems* **18** (1986) 105–118

[8] Kim, S. H., Ahn, B. S.: Interactive Group Decision Making Procedure Under Incomplete Information, *European Journal Of Operational Research* **116** (1999) 498–507

[9] Kim, S. H., Choi, S. H., Kim, J. K.: An interactive procedure for multiple attribute group decision making with in-

complete information: Range-based approach, *European Journal of Operational Research* **118** (1999) 139–152

[10] Orlovski, S. A.: Decision-making with fuzzy preference relations. *Fuzzy Sets and Systems* **1** (1978) 155–167

[11] Tanino, T.: Fuzzy preference orderings in group decision making. *Fuzzy Sets and Systems* **12** (1984) 117–131

[12] Tanino, T.: Fuzzy preference relations in group decision making. In: J. Kacprzyk, M. Roubens (Eds.). *Non-Conventional Preference Relations in Decision Making.* Springer-Verlag, Berlin, (1988) 54–71

[13] Xu, Z.S.: Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation. *International Journal of Approximate Reasoning*, **36:3** (2004) 261–270

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