Characterization of neighborhood operators for covering based rough sets, using duality and adjointness

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Abstract

Covering based Rough Sets are an important generalization of Rough Set Theory. Basically, they replace the partition generated from an equivalence relation by a covering. In this context many approximation operators can be defined [16, 26, 27, 28, 34]. In this paper we want to discover relationships among approximation operators defined from neighborhoods. We use the concepts of duality, adjointness and conjugacy to characterize the approximation operators. Moreover we establish an order relation for these approximation operators.

Keywords: rough sets, coverings, approximations, neighborhood operator, order relation.

1. Introduction

Rough Set Theory has been generalized from different perspectives. One generalization of rough sets is to replace the equivalence relation by a general binary relation. In this case, the binary relation determines collections of sets that no longer form a partition of the universal set U. This generalization has been used in applications with incomplete information systems and tables with continuous attributes [3, 4, 19]. A second generalization is to replace the partition obtained by the equivalence relation with a covering; i.e., a collection of nonempty sets with union equal to U. There are many works in these two directions and some connections between the two generalizations have been established, for example in [19, 32, 31, 35]. Order relations among different approximation operators are important

We are interested in approximation operators defined from a general neighborhood operator. According to [22] there are at least four neighborhood operators from a covering $\mathbb{C}$. The classification analysis in the rough set theory can be included into neighborhood system theory because for each object in the system, one or more classes are related to such object. For instance, in Pawlak's rough set, each object has an equivalence class; in covering approximation space, each object belongs to at least one block of the covering [17]. We want to establish relationships between the various definitions of approximation operators in the covering-based rough set model; and secondly, we want to evaluate the existing proposals with respect to the adjointness condition, providing in particular a characterization of approximation operators pairs that are both dual and adjoint. In this way, we hope to provide a clear cut roadmap for the covering-based rough set landscape, pinpointing the most useful operators among the many that have been proposed in the literature and guiding future research directions.

The remainder of this paper is organized as follows. Section 2 presents preliminary concepts about rough sets and lower and upper approximations in covering based rough sets, as well as the necessary lattice concepts about duality, conjugacy and adjointness. In Section 3, we present equivalences and relationships between various approximation operators, evaluate which of them satisfy adjointness, and end with the characterization theorem for dual and adjoint pairs. Section 4 presents an order relation among different approximation operators. Finally, Section 5 presents some conclusions and future work.

2. Preliminaries

2.1. Information Systems

An information system in the sense of Pawlak [8] is a 4-tuple $S = (U, \mathcal{A}, V, f)$ where U is a finite set of objects, A is a finite set of attributes, V is a function with domain: Dom $V = \mathcal{A}$, V_a is the set of values of $a \in \mathcal{A}$ and $f : U \to \Pi_{a\in\mathcal{A}} V_a$ is called the information function.

Each subset $P \subseteq \mathcal{A}$ of attributes determines an equivalence relation E_P on U, called the indiscernibility relation for P, and defined for $x, y \in U$ by:

$$xE_P y \Leftrightarrow \forall a \in P, (f(x)_a = f(y)_a). \tag{1}$$

A decision System is an information system with a distinguished attribute, which establishes a classification. In Table 1 the distinguished attribute is "*Class*". The classification in this case is the partition: {normal, sick} $= \{\{1,5,6\},\{2,3,4\}\}$.

Objects	Attributes			
Patient	a_1	a_2	a_3	**Class**
1	G	G	B	normal
2	B	B	B	sick
3	M	G	A	sick
4	M	B	B	sick
5	M	B	B	normal
6	G	B	A	normal

Table 1: A decision system.

2.2. Lower and upper approximations

The ordered pair (U,E), where U is a set and E is an equivalence relation is called an approximation space. U/E is the set of equivalence classes, called quotient set. In rough set theory the equivalence classes are used to define approximations of a set A.

Definition 1. *If $A \subseteq U$, we define a lower and an upper approximation of A, by means of:*

$$\underline{apr}(A) = \{x \in U : [x] \subseteq A\} \tag{2}$$
$$\overline{apr}(A) = \{x \in U : [x] \cap A \neq \emptyset\} \tag{3}$$

The difference between the upper and the lower approximations: $B(A) = \overline{apr}(A) - \underline{apr}(A)$, is called the boundary of A. If A is a subset of U such that $B(A) \neq \emptyset$, A is called a rough set.

Example 1. *From table 1, if we select the attributes $P = \{a_1, a_2, a_3\}$, the equivalence relation defines the partition:*

$$U/E = \{\{1\},\{2\},\{3\},\{4,5\},\{6\}\}.$$

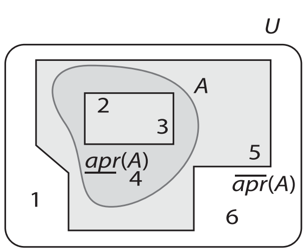

Fig. 1: Approximations of $A = 234$.

In figure 1 the set $A = \{2,3,4\}$ represents the concept "sick", according to partition, the lower approximation of A is $\underline{apr}(A) = \{2,3\}$ and the upper approximation is $\overline{apr}(A) = \{2,3,4,5\}$.

From the definition is easy to see that $\underline{apr}(A) \subseteq A \subseteq \overline{apr}(A)$.

2.3. Properties

The upper approximation operator $\overline{apr}$ satisfies the properties of table 2. Properties 3, 5 and 7 mean that

	Upper approximation
1	$\overline{apr}(U) = U$
2	$\overline{apr}(\emptyset) = \emptyset$
3	$X \subseteq \overline{apr}(X)$
4	$\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$
5	$\overline{apr}^2 = \overline{apr}$
6	$\overline{apr}(-X) = -\underline{apr}(X)$
7	$X \subseteq Y \Rightarrow \overline{apr}(X) \subseteq \overline{apr}(Y)$

Table 2: Properties of upper approximation.

$\overline{apr}$ is a closure operator, while properties 2 and 4 that $\overline{apr}$ is a topological closure operator [35]. Similar properties can be established for lower approximation. Property 6 establishes a duality relation between $\overline{apr}$ and $\overline{apr}$.

2.4. Covering based Rough Sets

We consider a generalization of Rough Sets where the partition is replaced by a covering of U.

Definition 2. *[26] Let $\mathbb{C} = \{K_i\}$ be a family of nonempty subsets of U. $\mathbb{C}$ is called a covering of U if $\cup K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.*

It is clear that a partition generated by an equivalence relation is a special case of a covering of U, so the concept of covering is a generalization of a partition.

In [22], Yao and Yao proposed a general framework for the study of covering based rough sets. It is based on the observation that when the partition U/E is generalized to a covering, the lower and upper approximations in Definition 1 are no longer equivalent. A distinguishing characteristic of their framework is the requirement that the obtained lower and upper approximation operators form a dual pair, that is, for $A \subseteq U$, $\underline{apr}(-A) = -\overline{apr}(A)$, where $-A$ represents the complement of A, i.e., $-A = U \setminus A$.

Below, we briefly review the generalizations of the element based definitions. In the element based definition, equivalence classes are replaced by neighborhood operators:

Definition 3. *[22] A neighborhood operator is a mapping $N : U \to \mathcal{P}(U)$. If $N(x) \neq \emptyset$ for all $x \in U$, N is called a serial neighborhood operator. If $x \in N(x)$ for all $x \in U$, N is called a reflexive neighborhood operator.*

Each neighborhood operator defines an ordered pair $(\overline{apr}_N, \underline{apr}_N)$ of dual approximation operators:

$$\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\} \tag{4}$$

$$\overline{apr}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\} \tag{5}$$

Different neighborhood operators, and hence different approximation operators in covering based rough sets, can be obtained from a covering $\mathbb{C}$.

Definition 4. *[22] If $\mathbb{C}$ is a covering of U and $x \in U$, a neighborhood system $\mathcal{C}(\mathbb{C}, x)$ is defined by:*

$$\mathcal{C}(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\} \tag{6}$$

In a neighborhood system $\mathcal{C}(\mathbb{C}, x)$, the minimal and maximal sets that contain an element $x \in U$ are particularly important.

Definition 5. *Let $(U, \mathbb{C})$ be a covering approximation space and x in U. The set of minimal elements in $\mathcal{C}(\mathbb{C}, x)$, ordered by inclusion, is called the minimal description of x, and denoted as $md(\mathbb{C}, x)$ [2]. On the other hand, the set of maximal elements in $\mathcal{C}(\mathbb{C}, x)$, ordered by inclusion, is called the maximal description of x, denoted as $MD(\mathbb{C}, x)$ [34].*

The sets $md(\mathbb{C}, x)$ and $MD(\mathbb{C}, x)$ represent extreme points of $\mathcal{C}(\mathbb{C}, x)$: for any $K \in \mathcal{C}(\mathbb{C}, x)$, we can find neighborhoods $K_1 \in md(\mathbb{C}, x)$ and $K_2 \in MD(\mathbb{C}, x)$ such that $K_1 \subseteq K \subseteq K_2$. From $md(\mathbb{C}, x)$ and $MD(\mathbb{C}, x)$, Yao and Yao [22] defined the following neighborhood operators:

1. $N_1(x) = \cap\{K : K \in md(\mathbb{C}, x)\}$
2. $N_2(x) = \cup\{K : K \in md(\mathbb{C}, x)\}$
3. $N_3(x) = \cap\{K : K \in MD(\mathbb{C}, x)\}$
4. $N_4(x) = \cup\{K : K \in MD(\mathbb{C}, x)\}$

For $i = 1, 2, 3, 4$, N_i is a reflexive neighborhood operator.

The set $N_1(x) = \cap md(\mathbb{C}, x)$ for each $x \in U$, is called the minimal neighborhood of x, and it satisfies some important properties as is shown in the following proposition:

Proposition 1. *[16] Let $\mathbb{C}$ be a covering of U and $K \in \mathbb{C}$, then*

- $K = \cup_{x \in K} N_1(x)$
- *If $y \in N_1(x)$ then $N_1(y) \subseteq N_1(x)$.*

The other neighborhood operators do not satisfy Proposition 1, as can be seen in the following example.

Example 2. *For simplicity, we use a special notation for sets and collections. For example, the set $\{1, 2, 3\}$ will be denoted by 123 and the collection $\{\{1, 2, 3\}, \{2, 3\}\}$ will be written as $\{123,\ 23\}$. Let us consider the covering $\mathbb{C} = \{1, 23, 123, 24, 234\}$ of $U = 1234$. The minimal and the maximal description $md(\mathbb{C}, x)$ and $MD(\mathbb{C}, x)$ are listed in Table 3.*

x	$md(\mathbb{C}, x)$	$MD(\mathbb{C}, x)$
1	{1}	{123}
2	{23, 24}	{123, 234}
3	{23}	{123, 234}
4	{24}	{234}

Table 3: Minimal and maximal description.

The four neighborhood operators obtained from $\mathcal{C}(\mathbb{C}, x)$ are listed in Table 4.

x	$N_1(x)$	$N_2(x)$	$N_3(x)$	$N_4(x)$
1	1	1	123	123
2	2	234	23	1234
3	23	23	23	1234
4	4	24	234	234

Table 4: Illustration of neighborhood operators.

For the set $A = 13$, we have that $\underline{apr}_{N_1}(A) = 1$, because $N_1(x) \subseteq A$ only for $x = 1$. $\underline{apr}_{N_2}(A) = 1$ and $\underline{apr}_{N_3}(A) = \underline{apr}_{N_4}(A) = \emptyset$. The upper approximations are: $\overline{apr}_{N_1}(A) = \overline{apr}_{N_2}(A) = 2346$, and $\overline{apr}_{N_3}(A) = \overline{apr}_{N_4}(A) = 1234$ then they do not satisfy Proposition 1.

We can see that $2 \in N_2(3)$, but $N_2(2) \nsubseteq N_2(3)$; also $3 \in N_4(4)$, but $N_4(3) \nsubseteq N_4(4)$.

2.5. Duality, conjugacy and Adjointness

In this section we present some basic notion about lattices, join and meet morphisms, conjugate and Galois connections. If L, K are lattices, a map $f : L \to K$ is a complete join morphism if whenever $S \subseteq L$ and $\vee S$ exists in L, then $\vee f(S)$ exists in K and $f(\vee S) = \vee f(S)$. Analogously, a map $f : L \to K$ is a complete meet morphism if whenever $S \subseteq L$ and $\wedge S$ exists in L, then $\wedge f(S)$ exists in K and $f(\wedge S) = \wedge f(S)$ [1].

A finite lattice is always complete, i.e. $\vee S$ and $\wedge S$ exist for all $S \subseteq L$. In this case a meet morphism f (a morphism that satisfies $f(a \wedge b) = f(a) \wedge f(b)$ for a and b in L) is a complete meet morphism, and dually, a join morphism f (a morphism that satisfies $f(a \vee b) = f(a) \vee f(b)$ for a and b in L) is a complete join morphism. Since in this paper, we assume that U is a finite universe, for the approximation operators we consider it will thus be sufficient to establish that they are meet (resp., join) morphisms.

Definition 6. *[5] Let f and g be two self-maps on a complete Boolean lattice B. We say that g is the dual* *of f, if for all $x \in B$,*

$$g(\sim x) = \sim f(x),$$

where $\sim x$ represents the complement of $x \in B$.

Definition 7. *[5] Let f and g be two self-maps on complete Boolean lattice B. We say that g is a conjugate of f, if for all $x, y \in B$,*

$$x \wedge f(y) = 0 \text{ if and only if } y \wedge g(x) = 0.$$

If g is a conjugate of f, then f is a conjugate of g. If a map f is the conjugate of itself, then f is called self-conjugate.

Definition 8. *[5] Let P and Q be two preorders; a pair (f, g) of maps $f : P \to Q$ and $g : Q \to P$ is called a Galois connection if*

$$f(p) \leq q \text{ if and only if } p \leq g(q) \tag{7}$$

The map g is called the *adjoint* of f and will be noted as f^a. The map f is called the *co-adjoint* of g and will be denoted as g_a.

Proposition 2. *[5] Let f be a self-map on a complete Boolean lattice B. Then f has a conjugate if and only if f is a complete join morphism on B.*

Proposition 3. *[5] Let B be a complete Boolean lattice. For any complete join morphism f on B, its adjoint is the dual of the conjugate of f. On the other hand, for any complete meet morphism g on B, its co-adjoint is the conjugate of the dual of g.*

3. Approximations from neighborhood operators

In this section $N : U \to \mathcal{P}(U)$ represents a reflexive neighborhood operator.

3.1. Lower and upper approximation

Let N be a neighborhood operator. We will consider two lower approximation, defined by:

- $L_1^N(A) = \{x : N(x) \subseteq A\}$
- $L_2^N(A) = \cup\{N(x) : N(x) \subseteq A\}$

We can note that L_1^N is the usual approximation operator defined in Equation 4, while L_2^N is the generalization of lower approximation $\underline{apr}$, used in granule based definition [22]. For upper approximation operators, we will consider the following :

- $G_5^N(A) = \cup\{N(x) : x \in A\}$
- $G_6^N(A) = \{x \in U : N(x) \cap A \neq \emptyset\}$
- $G_7^N(A) = \cup\{N(x) : N(x) \cap A \neq \emptyset\}$

This notation is a generalization of the pairs of approximation operators defined in [14, 15, 31], using a general neighborhood operator.

By definition, we know that $L_1^N = \underline{apr}_N$, and $G_6^N = \overline{apr}_N$, therefore L_1^N is the dual of G_6^N, i.e:

$$L_1^N(-A) = -G_6^N(A)$$

For the neighborhood operator N_1, we have the equality:

$$L_1^{N_1}(A) = L_2^{N_1}(A).$$

In general, $L_1^{N_i}(A) \neq L_2^{N_i}(A)$, for $i = 2, 3, 4$.

For establishing a relation among these approximation operators, we first establish an important conjugacy relation between the upper approximation operators G_5 and G_6. This relationship holds regardless of the neighborhood operator N which is used in the definition, so we begin by proving the following proposition.

Proposition 4. *If N a neighborhood operator, then G_5^N is the conjugate of G_6^N.*

Proof. We show that $A \cap G_5^N(B) \neq \emptyset$ if and only if $B \cap G_6^N(A) \neq \emptyset$, for $A, B \subseteq U$.

If $A \cap G_5^N(B) \neq \emptyset$, then there exists $w \in U$ such that $w \in A$ and $w \in G_5^N(B)$. Since $w \in G_5^N(B)$, there exists $x_0 \in B$ such that $w \in N(x_0)$. Then $N(x_0) \cap A \neq \emptyset$, with $x_0 \in G_6^N(B)$. Since $x_0 \in B$, then $B \cap G_6^N(A) \neq \emptyset$.

If $B \cap G_6^N(A) \neq \emptyset$, then there exists $w \in U$ such that $w \in B$ and $w \in G_6^N(A)$, i.e., $w \in B$ and $N(w) \cap A \neq \emptyset$. Then there exists z such that $z \in N(w)$ and $z \in A$. Since $z \in N(w)$ and $w \in B$, then $z \in G_5^N(B)$. So, $z \in A \cap G_5^N(B)$, with $A \cap G_5^N(B) \neq \emptyset$. ☐

Next, we prove that $L_2^{N_1}$ is the adjoint of $G_5^{N_1}$. For this, we need the following lemma.

Lemma 1. *For all $w \in U$, $G_5^{N_1}(N_1(w)) = N_1(w)$.*

Proof. By Proposition 1, from $x \in N_1(w)$ follows $N_1(x) \subseteq N_1(w)$, hence $G_5^{N_1}(N_1(w)) \subseteq N_1(w)$. On the other hand, it is clear that $N_1(w) \subseteq G_5^{N_1}(N_1(w))$, since $w \in N_1(w)$. ☐

This lemma is valid only for the N_1 neighborhood operator. For example, we can see that $G_5^{N_2}(N_2(3)) = G_5^{N_2}(23) = 234 \neq N_2(3)$ and $G_5^{N_4}(N_4(1)) = G_5^{N_4}(123) = 1234 \neq N_4(1)$.

Proposition 5. $L_2^{N_1} = (G_5^{N_1})^a$.

Proof. We will show that $L_2^{N_1}(A) \subseteq (G_5^{N_1})^a(A)$ and $(G_5^{N_1})^a(A) \subseteq L_2^{N_1}(A)$, for $A \subseteq U$. If $w \in L_2^{N_1}(A)$, there exists $x \in U$ such that $w \in N_1(x)$ with $N_1(x) \subseteq A$. The upper approximation $HG_5^{N_1}$ of $N_1(x)$ is equal to $N_1(x)$, by Lemma 1; i.e., $G_5^{N_1}(N_1(x)) = N_1(x)$.

Hence, $w \in \cup\{Y \subseteq U : G_5^{N_1}(Y) \subseteq A\}$, so $w \in (G_5^{N_1})^a(A)$. On the other hand, if $w \in (G_5^{N_1})^a(A)$, then there exists $Y \subseteq U$, such that $w \in Y$ and $G_5^{N_1}(Y) \subseteq A$; i.e., $\cup\{N_1(x) : x \in Y\} \subseteq A$; in particular, $w \in N_1(w) \subseteq G_5^{N_1}(Y) \subseteq A$, so $w \in L_2^{N_1}(A)$. □

Corollary 1. *The dual of $G_6^{N_1}$ is equal to $L_2^{N_1}$.*

According to Propositions 5 and 3, we have: $L_2^{N_1} = (G_5^{N_1})^a = ((G_5^{N_1})^c)^\partial = (G_6^{N_1})^\partial$.

Proposition 6. *$G_7^{N_1}$ is self-conjugate.*

Proof. According to Proposition 3 and the fact that $G_7^{N_1}$ is a join morphism, $(G_7^{N_1})^a = ((G_7^{N_1})^c)^\partial$, so $G_7^{N_1}$ is self-conjugate if and only if $(G_7^{N_1})^a = (G_7^{N_1})^\partial$, that is $(G_7^{N_1})^a(\sim A) = \sim G_7^{N_1}(A)$. We show that $(G_7^{N_1})^a(\sim A) = \sim G_7^{N_1}(A)$ for any $A \subseteq U$. $x \notin G_7^{N_1}(A)$ if and only if $N_1(x) \cap A = \emptyset$ if and only if $N_1(x) \subseteq \sim A$ if and only if $x \in (G_7^{N_1})^a(\sim A)$. □

3.2. Neighborhood systems from binary relations

Järvinen shows in [5] that there also exist Galois connections in generalized rough sets based on a binary relations. In particular, if R is a binary relation on U and $x \in U$, the sets:

$$\begin{aligned} R(x) &= \{y \in U : xRy\} \\ R^{-1}(x) &= \{y \in U : yRx\} \end{aligned} \tag{8}$$

are called successor and predecessor neighborhoods, respectively. The ordered pairs $(\overline{apr}_R, \underline{apr}_{R^{-1}})$ and $(\overline{apr}_{R^{-1}}, \underline{apr}_R)$, defined using the element based definitions (4) and (5), with $R(x)$ and $R^{-1}(x)$ instead of $N(x)$, form adjoint pairs. Moreover, $(\overline{apr}_R, \underline{apr}_R)$ and $(\overline{apr}_{R^{-1}}, \underline{apr}_R^{-1})$ are dual pairs.

On the other hand, Yao [19] established the following important proposition which relates dual pairs of approximation operators with the relation-based generalized rough set model considered by Järvinen.

Proposition 7. *[18] Suppose $(\overline{apr}, \underline{apr}) : \mathcal{P}(U) \to \mathcal{P}(U)$ is a dual pair of approximation operators, such that $\overline{apr}$ is a join morphism and $\overline{apr}(\emptyset) = \emptyset$. There exists a symmetric relation R on U, such that $\underline{apr}(A) = \underline{apr}_R(A)$ and $\overline{apr}(A) = \overline{apr}_R(A)$ for all $A \subseteq U$ if and only if the pair $(\overline{apr}, \underline{apr})$ satisfies: $A \subseteq \underline{apr}(\overline{apr}(A))$.*

By duality, we know that $\overline{apr}$ is a join morphism if and only if $\underline{apr}$ is a meet morphism and $\overline{apr}(\emptyset) = \emptyset$ if and only if $\underline{apr}(U) = U$. According to the proof, the symmetric relation R is defined by, for x, y in U,

$$xRy \Leftrightarrow x \in \overline{apr}(\{y\}) \tag{9}$$

We first establish that upper (resp., lower) approximation element based definitions have adjoints (resp., co-adjoints).

Proposition 8. *For any neighborhood operator N, $\underline{apr}_N$ is a meet morphism.*

Proof. Since $\underline{apr}_N(A) = \{x \in U : N(x) \subseteq A\}$, we have $x \in \underline{apr}_N(A \cap B)$ iff $N(x) \subseteq A \cap B$ iff $N(x) \subseteq A$ and $N(x) \subseteq B$ iff $x \in \underline{apr}_N(A)$ and $x \in \underline{apr}_N(B)$ iff $x \in \underline{apr}_N(A) \cap \underline{apr}_N(B)$. □

Corollary 2. *For any neighborhood operator N, $\overline{apr}_N$ is a join morphism.*

Corollary 3. *For any neighborhood operator N, $\underline{apr}_N$ has a co-adjoint and it is equal to the conjugate of $\overline{apr}_N$.*

By Proposition 3 and the duality of $\overline{apr}_N$ and $\underline{apr}_N$, $\left(\underline{apr}_N\right)_a = \left(\underline{apr}_N^\partial\right)^c = (\overline{apr}_N)^c = (G_6^N)^c = G_5^N$.

Corollary 4. *For any neighborhood operator N, $\overline{apr}_N$ has an adjoint and it is equal to the dual of G_5^N.*

Indeed, by Proposition 3, we find $(\overline{apr}_N)^a = ((\overline{apr}_N)^c)^\partial = \left(G_5^N\right)^\partial$.

The remaining question now is whether $(\overline{apr}_N, \underline{apr}_N)$ can ever form an adjoint pair. For this to hold, based on the above we need to have that $\left(\underline{apr}_N\right)_a = G_5^N = G_6^N = \overline{apr}_N$.

Proposition 9. *$(\overline{apr}_N, \underline{apr}_N)$ is an adjoint pair if and only if N satisfies $G_5^N = G_6^N$.*

The following proposition characterizes the neighborhood operators N that satisfy $G_5^N = G_6^N$, and establishes the link with the generalized rough set model based on a binary relation.

Proposition 10. *Let N be a neighborhood operator. The following are equivalent:*

1. *For all x, y in U, N satisfies*
$$y \in N(x) \Rightarrow x \in N(y) \tag{10}$$
2. *$G_5^N = G_6^N$*
3. *There exists a symmetric binary relation R on U such that $N(x) = \{y \in U : xRy\}$.*

Proof. We first prove (i) ⇒ (ii). Let $A \subseteq U$. If $w \in G_5^N(A)$, then $w \in \cup\{N(x) : x \in A\}$. This means that $w \in N(x)$ for some $x \in A$, and by (10) $x \in N(w)$, so $N(w) \cap A \neq \emptyset$. Hence $w \in G_6^N(A)$.

If $w \in G_6^N(A)$, then $N(w) \cap A \neq \emptyset$. In other words, there exists $x \in U$ with $x \in A$ and $x \in N(w)$. By (10), $w \in N(x)$ and thus $w \in \cup\{N(x) : x \in A\} = G_5^N(A)$.

On the other hand, to prove (ii) ⇒ (i), by the definition of G_5^N, we have $G_5^N(\{x\}) = N(x)$. If $G_5^N(A) = G_6^N(A)$, for all $A \subseteq U$ and $y \in N(x)$ then $N(x) \cap \{y\} \neq \emptyset$, so $x \in G_6^N(\{y\}) = G_5^N(\{y\}) = N(y)$.

Finally, the equivalence (i) ⇔ (iii) is immediate, with R defined by $xRy \Leftrightarrow x \in N(y)$ for x, y in U. □

The proposition thus shows that the only adjoint pairs among element-based definitions are those for which the neighborhood is defined by Eq. (8), with symmetric R.

4. Order relation for neighborhood operators

We establish the relationship among approximation operators defined in equations (4) and (5) from the neighborhood systems using the following propositions.

Proposition 11. *If N and N' are neighborhood operators such that $N(x) \subseteq N'(x)$ for all $x \in U$, then $\underline{apr}_N \geq \underline{apr}_{N'}$ and $\overline{apr}_N \leq \overline{apr}_{N'}$.*

Proof. If $x \in \underline{apr}_{N'}(A)$, $N'(x) \subseteq A$, but $N(x) \subseteq N'(x) \subseteq A$ for all $x \in U$, so $x \in \underline{apr}_N(A)$. The other relation can be proved similarly. □

From the definition of neighborhood system it is easy to show that $N_1(x) \subseteq N_2(x)$ and $N_3(x) \subseteq N_4(x)$, so $\underline{apr}_{N_1} \geq \underline{apr}_{N_2}$ and $\overline{apr}_{N_3} \leq \overline{apr}_{N_4}$. We also have the following result:

Proposition 12. *For $x \in U$, it holds that $N_1(x) \subseteq N_3(x)$ and $N_2(x) \subseteq N_4(x)$.*

Proof. For $N_1(x) \subseteq N_3(x)$, we can see that for each $K \in N_1(x)$ there exists $K' \in N_3(x)$ such that $K \subseteq K'$ and vice versa. So, $\cap\{K \in md(\mathbb{C}, x)\} \subseteq \cap\{K' \in MD(\mathbb{C}, x)\}$ from which follows $N_1(x) \subseteq N_3(x)$. The other relation can be proved similarly. □

From propositions 11 and 12 we have the order relation between the approximation operators: $L_1^{N_4} \leq L_1^{N_2} \leq L_1^{N_1}$ and $L_1^{N_4} \leq L_1^{N_3} \leq L_1^{N_1}$. The order relations for upper and lower approximation operators, defined by order "$\leq$" is shown in Figure 2.

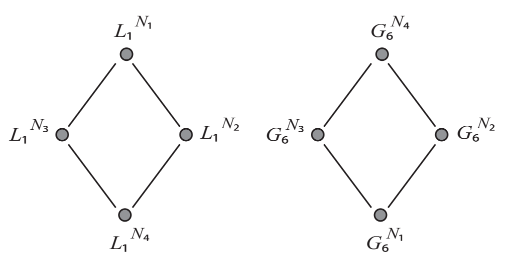

Fig. 2: Order relation for neighborhood based approximations

By definition of upper approximation operators it is easy to show that: $G_6^N(A) \subseteq G_7^N(A)$ and $G_6^N(A) \subseteq G_7^N(A)$ for every $A \subseteq U$. On the other hand G_5^N and G_6^N are not comparable, so we have the order depicted in Figure 3.

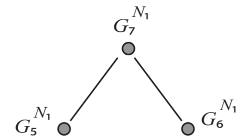

Fig. 3: Order relation for neighborhood based approximation operators

5. Conclusions

In this paper, we have studied relationships between approximation operators defined from neighborhood operators within the covering-based rough set model. We have also demonstrated that $(G_5^{N_1}, L_2^{N_1})$ is an adjoint, non-dual pair, while $(G_6^{N_1}, L_2^{N_1})$ is a dual, non adjoint pair. We have established a characterization of pairs of dual approximation operators based in neighborhoods, using adjointness.

We have also established an order relation among approximation operators. As future work, will be interesting to study similar properties with other approximation operators for covering based rough sets.

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